A Simple Solution to a Bandwidth Pricing Anomaly

MARTIN I. REIMAN WIM SWELDENS

Bell Laboratories, Lucent Technologies
600 Mountain Avenue, Murray Hill, NJ 07974
{marty,wim}@lucent.com

February, 2001

Abstract

We show that it is much safer to compute bandwidth spot prices from forward prices than vice versa. Our observation eliminates a geographic arbitrage opportunity in forward prices pointed out by Chiu and Crametz.

1 Introduction

There is a current trend towards turning bandwidth into a commodity, with standardized contracts traded on exchanges. Hence it is important to understand pricing relationships for bandwidth. Chiu and Crametz [1] examine two 'no-arbitrage' relationships that must hold in a liquid bandwidth market: temporal and geographical. Temporal arbitrage occurs when the price of a particular spot contract is not equal to the price of an equivalent contract constructed using forwards contracts. Geographical arbitrage occurs when the sum of prices on the links of an indirect path connecting two cities is less than the price of the link that directly joins them. If either of these situations were to occur in a liquid enough market, an astute trader could make riskless profits.

Through an illuminating example, Chiu and Crametz [1] show that it is possible to have spot prices that satisfy the no-geographical arbitrage conditions, while the associated forward prices implied by the no-temporal arbitrage condition allow geographical arbitrage. This example raises the possibility that the absence of arbitrage opportunities in a realistic network may be hard to verify. We show that the apparent pricing anomaly can be avoided in a simple manner. In [1] spot prices are taken as basic data and forward prices are derived from them. We observe that this relationship is numerically unstable and instead propose to take forward prices as basic data and derive spot prices from the forward prices. The key point, which we show below, is that if forward prices satisfy the no-geographical arbitrage conditions, then the spot prices derived from them are also guaranteed to satisfy the no-geographical arbitrage condition. Thus, our approach avoids the pricing anomaly.

2 Basic relationship between spot and forward prices

We follow the notation and terminology in [1]. A spot bandwidth contract is specified in terms of the city pair connected, the contract duration, and the price. (Other important characteristics, such as throughput and quality-of-service guarantees are assumed identical for all contracts we consider.) Today most contracts are in one-month periods, but there is a trend towards shorter time periods. Therefore we consider a general time period unit in our analysis. The current (about to begin) time period is indexed 0, and the spot price for a one-period contract is p_0. A two-period contract has an associated payment stream (p_1, p_1), representing a payment of p_1 in each of the two periods of the contract. Similarly, an $n + 1$ period contract has a payment stream $(p_n, p_n, \ldots, p_n)$, see Figure 1.

$$
\begin{array}{ccccc}
p_0 & & & & \\
p_1 & p_1 & & & \\
p_2 & p_2 & p_2 & & \\
\vdots & & & \ddots & \\
p_n & p_n & \cdots & \cdots & p_n
\end{array}
\qquad\qquad
\begin{array}{ccccc}
f_0 & & & & \\
& f_1 & & & \\
& & f_2 & & \\
& & & \ddots & \\
& & & & f_n
\end{array}
$$

Figure 1: Structure of spot and forward bandwidth prices.

A forward bandwidth contract, for a particular city pair, specifies delivery of one period of capacity n periods from now, for a payment in the period of delivery of f_n, with the current period again indexed by 0.

The existence of both spot and forward contracts provides more than one way to acquire bandwidth for the $n + 1$ periods $0, 1, \ldots, n$. If the total (discounted) prices paid for these different contracts are not equal, a 'frictionless market' would present an opportunity for arbitrage. This immediately implies that $p_0 = f_0$, since the current period spot and forward contracts are identical.

Let d_{i0} denote the discount factor from period i to period 0. Then the no-temporal arbitrage condition implies that

$$f_0 + d_{10} f_1 + d_{20} f_2 + \cdots + d_{n0} f_n = p_n + d_{10} p_n + d_{20} p_n + \cdots + d_{n0} p_n = ds_{n0} p_n, \tag{1}$$

where $ds_{n0} = d_{00} + d_{10} + \cdots + d_{n0}$. From this one can immediately derive that [1],

$$p_n = \frac{1}{ds_{n0}} \sum_{i=0}^{n} d_{i0}\, f_i \,. \tag{2}$$

Using (1) we see that $ds_{n-1,0} p_{n-1} + d_{n0} f_n = ds_{n0} p_n$, from which it follows that [1],

$$f_n = \frac{ds_{n0}\, p_n - ds_{n-1,0}\, p_{n-1}}{d_{n0}}. \tag{3}$$

Equations (2) and (3) represent the essence of no-temporal arbitrage: given spot prices $p_0, p_1, \ldots, p_n$, the forward prices $f_0, f_1, \ldots, f_n$ are uniquely determined by (3), and given forward prices $f_0, f_1, \ldots, f_n$, the spot prices $p_0, p_1, \ldots, p_n$ are uniquely determined by (2).

3 Sensitivity analysis of bandwidth prices

We next examine the numerical stability of both equations (2) and (3). We want to understand how fluctuations in spot prices affect forward prices and vice versa.

3.1 Computing forward prices from spot prices

We first examine equation (3) under the simplifying assumption that there is no discount, i.e, $d_{i0} = 1$. Then

$$f_n = np_n - (n-1)p_{n-1}.$$

The forward price f_n is computed as the difference between a n-period contract (np_n) and a $(n-1)$-period contract $(n-1)p_{n-1}$. For large n, p_n and p_{n-1} will be close and we take the difference of two large, almost equal numbers. This is the canonical example of an numerically unstable operation [2, Section 1.2-1.3]. Consider the example where $n = 30$, $p_{30} = 1$, and $p_{29} = 1.01$. Then $f_{30} = 30 \times 1 - 29 \times 1.01 = 0.71$. Next consider a 1% fluctuation where $p_{30} = 1.01$ and $p_{29} = 1$. Then $f_{30} = 1.30$, an 83% fluctuation! The difference formula has the effect of magnifying the fluctuations.

This becomes clear when we do a sensitivity analysis. Say p_n and p_{n-1} have a relative fluctuation of ϵ, or $\Delta p_n \leqslant \epsilon p_n$ and $\Delta p_{n-1} \leqslant \epsilon p_{n-1}$. Then

$$\frac{\Delta f_n}{f_n} \leqslant \frac{n\Delta p_n + (n-1)\Delta p_{n-1}}{f_n} \leqslant \frac{n\epsilon p_n + (n-1)\epsilon p_{n-1}}{f_n} \approx 2n\epsilon.$$

The last approximation is based on the fact that p_n, p_{n-1}, and f_n are typically of the same order of magnitude. This means that fluctuations get multiplied approximately with a factor $2n$. This gets worse when the standard contract time becomes shorter and n becomes larger.

We next look at the case of a constant discount factor, i.e, $d_{i0} = d^i$. Then

$$ds_{n0} = \sum_{i=0}^{n} d^i = \frac{1-d^{n+1}}{1-d},$$

and

$$f_n = \frac{(1-d^{n+1})p_n - (1-d^n)p_{n-1}}{(1-d)d^n}. \tag{4}$$

Consider the numerical example above with $d = 0.995$ (corresponding to an annual interest rate of approximately 6% if the period is a month). With $p_{30} = 1$, and $p_{29} = 1.01$, $f_{30} = 0.675$. With $p_{30} = 1.01$ and $p_{29} = 1$, $f_{30} = 1.33$, a 98% fluctuation. So discounting increases the fluctuation. In the fully general case of equation (3), the same magnifying effect occurs.

3.2 Computing spot prices from forward prices

From equation (2) we see that a spot price p_n is computed as convex combination of forward prices f_i: each coefficient d_{i0}/ds_{n0} is positive and all coefficients sum to one. Given that there is no subtraction, convex combinations are the most stable computations possible. Now if the forward prices carry a relative fluctuation of ϵ, $\Delta f_n \leqslant \epsilon f_n$ then the fluctuation on the spot prices is

$$\Delta p_n \leqslant \frac{1}{ds_{n0}} \sum_{i=1}^{n} d_{i0} \Delta f_i \leqslant \frac{\epsilon}{ds_{n0}} \sum_{i=1}^{n} d_{i0} f_i = \epsilon p_n.$$

Thus the relative fluctuation on the spot prices is ϵ as well and there is no magnification. If all the forward prices fluctuate by 1%, the spot price can change no more than 1%. If the fluctuations are independent, then the law of large numbers makes the fluctuation on the spot price much less than 1%.

4 No-geographical arbitrage

To illustrate the no-geographical arbitrage condition, a three-link triangular network, consisting of links a, b, c, is considered in [1]. Let p_n^k and f_n^k denote, respectively, spot and forward prices on link k for $k = a, b$, or c. The no-geographical arbitrage condition is specified in [1] by the three 'triangle inequalities'

$$p_n^b + p_n^c \geqslant p_n^a, \qquad p_n^a + p_n^c \geqslant p_n^b, \qquad p_n^a + p_n^b \geqslant p_n^c\,. \tag{5}$$

To understand the arbitrage possibilities, note that if the first inequality is not satisfied a trader can buy capacity on links b and c and sell capacity on link a, pocketing the difference $p_n^a - (p_n^b + p_n^c) > 0$. (There is an implicit assumption here that the two-link route meets quality-of-service requirements.) Similar inequalities should hold for the forward prices:

$$f_n^b + f_n^c \geqslant f_n^a, \qquad f_n^a + f_n^c \geqslant f_n^b, \qquad f_n^a + f_n^b \geqslant f_n^c\,. \tag{6}$$

Through an example, Chiu and Crametz [1] show that it is possible to find spot prices that satisfy the triangle inequalities (5), such that the associated forward prices obtained from these spot prices via (3) do not satisfy the triangle inequalities (6). This illuminating example is also alarming because it indicates that finding no-arbitrage conditions for a network may be a daunting task. However, as we now show, a simple solution to this pricing anomaly is available: start with forward prices rather than spot prices. The mapping from forward prices to spot prices is numerically better behaved than the mapping taking spot prices into forward prices. Thus if we consider forward prices that satisfy triangle inequalities (6) and compute spot prices from then, then we show that that the spot prices have to satisfy the triangle inequalities (5). This assures adherence to the no-geographical arbitrage condition for spot prices obtained from forward prices that satisfy the no-geographical arbitrage condition.

From equation (2), the lack of temporal arbitrage implies that

$$p_n^k = \frac{1}{ds_{n0}} \sum_{i=0}^{n} d_{i0}\, f_i^k \tag{7}$$

for any n and $k = a, b, c$. It is now straightforward to show that there is no geographical arbitrage for the spot prices. Note that

$$ds_{n0}(p_n^a + p_n^b) = \sum_{i=0}^{n} d_{i0}\, f_i^a + \sum_{i=0}^{n} d_{i0}\, f_i^b = \sum_{i=0}^{n} d_{i0}\, (f_i^a + f_i^b) \geqslant \sum_{i=0}^{n} d_{i0}\, f_i^c = ds_{n0}\, p_n^c, \tag{8}$$

where the inequality follows from (6) and the final equality follows from (7). Dividing by the positive quantity ds_{n0} gives

$$p_n^a + b_n^b \geqslant p_n^c. \tag{9}$$

Similar manipulations, mutatis mutandis, show that $p_n^a + p_n^c \geqslant p_n^b$ and $p_n^b + p_n^c \geqslant p_n^a$.

5 Conclusions

We have given two reasons why it is better to compute bandwidth spot prices from forward prices than forward prices from spot prices. First, this is a much more stable computation which does not magnify fluctuations and second, it eliminates a geographical arbitrage opportunity.

Needless to say, the network and trading model used in this paper is unrealistically simplistic compared to actual telecommunication networks and bandwidth exchanges. Many interesting questions remain. Our model assumes a liquid market and ignores any external information that may influence future prices. How do we take this external information into account? The triangular network is only the simplest geographics arbitrage opportunity. What can we say about no-geographical arbitrage conditions in more general networks? The complete specification of no-geographical arbitrage conditions is beyond the scope of this short note. What can be said is that these conditions will be given by inequalities, and any set of inequalities that holds for forward prices can be easily shown to hold for spot prices as well.

Reference

[1] S. Chiu and J. P. Crametz. Surprising pricing relationships. *Bandwidth Special Report*, pages 12–14, July 2000.

[2] J. Stoer and R. Bulirsch. *Introduction to Numerical Analysis*. Springer Verlag, New York, 1980.