A NEW CLASS OF UNBALANCED HAAR WAVELETS THAT FORM AN UNCONDITIONAL BASIS FOR L_p ON GENERAL MEASURE SPACES

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ABSTRACT. Given a complete separable σ-finite measure space (X, Σ, μ) and nested partitions of X, we construct unbalanced Haar-like wavelets on X that form an unconditional basis for $\mathrm{L}_p(X, \Sigma, \mu)$ where $1 < p < \infty$. Our construction and proofs build upon ideas of Burkholder and Mitrea. We show that if (X, Σ, μ) is not purely atomic, then the unconditional basis constant of our basis is $(\max(p, q) - 1)$. We derive a fast algorithm to compute the coefficients.

1. INTRODUCTION

Our goal is, given a measure space (X, Σ, μ) and nested partitions of X, to construct unbalanced Haar-like wavelets on X that form an unconditional basis for $\mathrm{L}_p \equiv \mathrm{L}_p(X, \Sigma, \mu)$ where $1 < p < \infty$.

Wavelets are traditionally defined on Euclidean spaces. They usually are the translates and dilates of one particular function and are orthogonal or biorthogonal with respect to the Lebesgue measure.

However, we work on a general measure space, which need not even have a vector space structure, so translation and dilation becomes void. Although our wavelets are not the translates and dilates of one function, they still enjoy the desirable properties of traditional wavelets, such as a multiresolution structure and an associated fast transform algorithm. Our setting allows for non-translation invariant measures and covers general nested partitions of arbitrary subsets of Euclidean spaces. Thus our wavelets are particularly useful in practical applications.

Our construction is inspired by and generalizes the construction [1, 15] of Mitrea wavelets on dyadic cubes in $\mathbf{R}^n$. Mitrea wavelets can be seen as a generalization of the unbalanced Haar wavelets introduced for non-translation invariant measures in [9].

To show that wavelets form an unconditional basis of L_p, one often uses Calderón-Zygmund theory and an interpolation result of Marcinkiewicz. We follow a different approach; we show that the wavelets essentially are a martingale difference sequence and thus are able to use Burkholder's celebrated inequality [4, 5, 6] to show that they form an unconditional basic sequence. This approach gives the best unconditional basis constant. We also show that in some cases the wavelets form a monotone basis.

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One aim of this paper is to illustrate how techniques from martingale and Banach space theory can be used in wavelet theory.

The paper is organized as follows. In Section 2 we set some notation and recall some classical results. In Section 3 we introduce the notion of a forest, which we use as an indexing set. We use the forest to define partitions in Section 4. Section 5 contains the construction of the wavelets while Section 6 contains the proof that they form an unconditional basis. We discuss the dual basis and a characterization in Section 7. The next two sections contain more practical results. Sections 8 shows the connection with multiresolution analysis and the fast wavelet transform, while Section 9 discusses the setting inside a Euclidean space.

2. Notation and Terminology and Basics

Recall that a countable family $\{\psi_\gamma\}_{\gamma\in\mathcal{G}}$ is an unconditional basis for L_p if for each $f\in\mathrm{L}_p$ there is a unique family $\{c_\gamma\}_{\gamma\in\mathcal{G}}$ of real numbers so that $\sum c_\gamma\psi_\gamma$ converges unconditionally to f in L_p-norm. This is the case if and only if the following two conditions hold:

(C1) $\operatorname{clos}\operatorname{span}\{\psi_\gamma \mid \gamma\in\mathcal{G}\} = \mathrm{L}_p$

(C2) a constant K exists so that for all finite subsets $\Gamma\subset\mathcal{G}$

$$\left\|\sum_{\gamma\in\Gamma}\epsilon_\gamma\, c_\gamma\, \psi_\gamma\right\|_p \leqslant K\,\left\|\sum_{\gamma\in\Gamma} c_\gamma\, \psi_\gamma\right\|_p ,$$

for all choices of $c_\gamma\in\mathbf{R}$ and $\epsilon_\gamma=\pm1$.

The smallest K for which condition (C2) holds, denoted $K_p(\{\psi_\gamma\})$, is the unconditional basis constant of $\{\psi_\gamma\}$.

Clearly, any Banach space with a countable basis is separable. If $1<p<\infty$, then a separable $\mathrm{L}_p(\mu)$ space has an unconditional basis [18, 8]. Pełczyński [19] showed that, for any finite non-purely atomic measure μ, the space $\mathrm{L}_1(\mu)$ does not even embed into a Banach space with an unconditional basis. Thus we restrict our attention to *separable* L_p spaces with $1<p<\infty$. In this setting, we know $\mathrm{L}_p(X,\Sigma,\mu)$ up to an isometric isomorphism. Recall that two Banach spaces E and F are isometrically isomorphic if there is an invertible bounded linear operator $T:E\to F$ so that $\|T\|=1=\left\|T^{-1}\right\|$. A separable $\mathrm{L}_p(X,\Sigma,\mu)$ is isometrically isomorphic to precisely one of the following spaces, where $\mathrm{L}_p(m)$ is the L_p space on the Lebesgue measure space on $[0,1]$: ℓ_p, ℓ_p^n, $\mathrm{L}_p(m)$, $\mathrm{L}_p(m)\oplus_p\ell_p$, $\mathrm{L}_p(m)\oplus_p\ell_p^n$ for some $n\in\mathbf{N}$ (cf. [23, Proposition III.A.1]). The isometric isomorphism basically follows from mapping (X,Σ,μ) into a combination of the Lebesgue measure space on $[0,1]$ and the counting measure space on $\mathbf{N}$. For practical reasons, we prefer to constructively build our wavelets directly on X instead of calling upon this mapping.

Throughout this paper, (X,Σ,μ) is a fixed complete measure space with μ taking values in the non-negative extended real numbers. Let Σ^+ be the collection of all sets in Σ with strictly positive, but finite, μ-measure; let $\widetilde{\Sigma}$ be any sub-σ-field of Σ such that the μ-completion of $(X,\widetilde{\Sigma})$ is (X,Σ). The support of a function $f : X \to \mathbf{R}$ is the set $\operatorname{supp} f \equiv \{x \in X \mid f(x) \neq 0\}$. For an arbitrary set S, let $\mathcal{P}(S)$ be the power set of S and $\#S$ be the cardinality of S. For $K \subset \mathcal{P}(S)$, let $\sigma(K)$ be the smallest σ-algebra containing K. For a function f on S, we follow the common practice of also denoting by f the natural extension of the original f to $\mathcal{P}(S)$. Throughout this paper, $1 < p < \infty$ is a fixed number with conjugate exponent q where $1/p + 1/q = 1$. Let $p^* = \max(p, q)$. The dual space L_p^* of $\mathrm{L}_p(X, \Sigma, \mu)$ is isometrically isomorphic to $\mathrm{L}_q(X, \Sigma, \mu)$, where $g \in \mathrm{L}_q(X, \Sigma, \mu)$ is identified with $x_g^* \in \mathrm{L}_p^*$ by

$$x_g^*(f) = \langle f, g \rangle = \int_X f\, g\, d\mu\ .$$

We say that $f \in \mathrm{L}_p$ is orthogonal to $g \in \mathrm{L}_q$ if $\langle f, g \rangle = 0$.

3. Trees and Forests

We formulate the notation of a forest, which is a useful indexing set. A *forest* $(\mathcal{F}, g, p, C, <)$ consists of a *countable* set $\mathcal{F}$, which has a (possibly empty) subset $\mathcal{R}$ of *root* elements, along with a generation function $g : \mathcal{F} \to \mathbf{Z}$, a parent function $p : \mathcal{F} \setminus \mathcal{R} \to \mathcal{F}$, a children function $C : \mathcal{F} \to \mathcal{P}(\mathcal{F})$, and an age partial ordering $<$ on $\mathcal{F}$, all of which satisfy the following properties:

(F1) $C(\alpha) = \{\beta \in \mathcal{F} \mid p(\beta) = \alpha\}$,
(F2) $0 \leqslant \#C(\alpha) < \infty$ for each $\alpha \in \mathcal{F}$,
(F3) if $\beta \in C(\alpha)$ then $g(\beta) = 1 + g(\alpha)$,
(F4) the ordering $<$ linearly orders $C(\alpha)$ for each $\alpha \in \mathcal{F}$.
(F5) if $g(\alpha) < g(\beta)$ and $p^n(\alpha) = p^m(\beta)$ for some $n, m \geqslant 0$, then $\beta < \alpha$,

where the power functions p^n of the parent function p are defined by p^0 being the identity function and $p^n(\alpha) = p(p^{n-1}(\alpha))$. If confusion is unlikely, we denote a forest $(\mathcal{F}, g, p, C, <)$ by just $\mathcal{F}$. The given partial ordering extends to a linear ordering of the whole forest with (F4) and (F5) still holding: it is only needed to extend the ordering as so to linearly order each k^{th}-generation $\mathcal{F}_k$ of $\mathcal{F}$ where

$$\mathcal{F}_k = \{\alpha \in \mathcal{F} \mid g(\alpha) = k\}\ .$$

Thus, henceforth, forests satisfy the additional property

(F6) the ordering $<$ linearly orders the the whole forest.

One thinks of a parent element $\alpha \in \mathcal{F}_k$ on the k^{th}-generation of $\mathcal{F}$ as spawning the children elements β with $\beta \in C(\alpha) \subset \mathcal{F}_{k+1}$. Root elements are denoted by ρ and have no parent. A forest $\mathcal{F}$ that satisfies the additional property

(T1) if $\alpha, \beta \in \mathcal{F}$, then there are $n, m \geqslant 0$ so that $p^n(\alpha) = p^m(\beta)$

is called a *tree*. A tree has at most one root element; a *rooted* tree has exactly one root.

A *leaf* is an element that has no children. Let $\mathcal{L}$ be the set of leaves in $\mathcal{F}$. On occasions it is convenient to think of a leaf as repeating itself in the later generations, for this consider

$$\mathcal{F}_k^* \; = \; \mathcal{F}_k \quad \bigcup \quad \left[\bigcup_{j<k} \quad \mathcal{L} \cap \mathcal{F}_j \right] \; .$$

Let

$$C_l(\alpha) = \begin{cases} C(\alpha) & \text{if } \alpha \notin \mathcal{L} \\ \{\alpha\} & \text{if } \alpha \in \mathcal{L}\, . \end{cases}$$

Define the power functions C^n (resp. $C_l^n(\alpha)$) of the function C (resp. $C_l(\alpha)$) analogous to the power functions p^n. Note that for $n \in \mathbf{N}$,

$$C_l^n(\alpha) \; = \; C^n(\alpha) \quad \bigcup \quad \bigcup_{j=0}^{n-1} \left[\, C^j(\alpha) \cap \mathcal{L} \,\right] \; .$$

A countable union of disjoint trees is a forest. Conversely, any forest $(\mathcal{F}, g, p, C, <)$ can be expressed as a countable union of disjoint trees. To see this, consider the equivalence relation $\sim$ on $\mathcal{F}$ given by $\alpha \sim \beta$ if and only if (T1) holds. This relation induces a partition of $\mathcal{F}$

$$\mathcal{F} = \bigcup_{\varkappa \in \mathcal{K}} \mathcal{F}(\varkappa) \tag{1}$$

into *disjoint* equivalence classes $\mathcal{F}(\varkappa)$ where the indexing set $\mathcal{K}$ is the induced quotient space. Each $\mathcal{F}(\varkappa)$ is a tree.

The concept of a forest, which is fairly technical, is introduced to help simplify the construction of wavelets from nested partitionings of X. Later we will reduce the general forest setting to three canonical cases of trees.

4. Partitions

We call a collection $\{X_\alpha \mid \alpha \in \mathcal{F}\}$ from Σ^+ a *nested partitioning* for X, with respect to the forest $\mathcal{F}$, if it satisfies the following partition properties:

(P1) $X_{\alpha_1} \cap X_{\alpha_2} = \emptyset \quad$ if $g(\alpha_1) = g(\alpha_2)$ and $\alpha_1 \neq \alpha_2$,

(P2) $X_\alpha \cap X_\rho = \emptyset \quad$ if $\rho \in \mathcal{R}$ and $p^n(\alpha) \neq \rho$ for each $n \geqslant 0$,

(P3) if X_α is not a leaf, then it can be written as the disjoint union

$$X_\alpha = \bigcup_{\beta \in C(\alpha)} X_\beta \; ,$$

(P4) $X = \cup\{X_\alpha \mid \alpha \in \mathcal{F}\}$,

(P5) $\sigma(\{X_\alpha \mid \alpha \in \mathcal{F}\}) = \widetilde{\Sigma}$.

The partitioning (1) of the forest into trees provides a partition of X. For each $\varkappa \in \mathcal{K}$, let

$$X(\varkappa) \;=\; \bigcup_{\alpha \in \mathcal{F}(\varkappa)} X_\alpha \;.$$

From the first three partition properties it follows that if $\varkappa_1 \neq \varkappa_2$ then $X(\varkappa_1)$ and $X(\varkappa_2)$ are disjoint. Thus X can be written as the *disjoint* union

$$X = \bigcup_{\varkappa \in \mathcal{K}} X(\varkappa) \;. \tag{2}$$

For each $\varkappa \in \mathcal{K}$, the subcollection $\{X_\alpha \mid \alpha \in \mathcal{F}(\varkappa)\}$ is a nested partitioning for $X(\varkappa)$ with respect to the tree $\mathcal{F}(\varkappa)$. The partitions

$$\pi_k(\varkappa) = \{X_\alpha \mid \alpha \in \mathcal{F}_k^* \cap \mathcal{F}(\varkappa)\}$$

of $X(\varkappa)$ are nested for $k \in g(\mathcal{F}(\varkappa))$. We will use the subcollection $\{X_\alpha \mid \alpha \in \mathcal{F}(\varkappa)\}$ to build wavelets on $X(\varkappa)$. Our wavelets Ξ will then be the union of the wavelets on each $X(\varkappa)$. Thus, for the time being, we will work with trees instead of forests. There are three types of nested partitionings of X with respect to a tree $\mathcal{T}$:

- Type I: $\mathcal{R} \neq \emptyset$ and thus $\mu(X) < \infty$,
- Type II: $\mu(X) < \infty$ and $\mathcal{R} = \emptyset$,
- Type III: $\mu(X) = \infty$ and thus $\mathcal{R} = \emptyset$.

Each type is handled slightly different. But before passing to the construction of the wavelets, we clarify the above notations with the following examples.

Example 1. Let (X, Σ, μ) be the Lebesgue measure space on $X = [0, 1)$. Consider the Type I tree $(\mathcal{T}, g, p, <)$ where

1. $\mathcal{T} = \{(n, k) \mid n = 0, 1, \ldots$ and $1 \leqslant k \leqslant 2^n\}$,
2. $\rho = (0, 1)$,
3. $g((n, k)) = n$,
4. $C((n, k)) = \{(n + 1, 2k - 1), (n + 1, 2k)\}$,
5. $(n + 1, 2k - 1) < (n + 1, 2k)$.

Let $X_{(n,k)} = [\, 2^{-n}\,(k - 1)\;,\; 2^{-n}\,(k)\,)$.

Example 2. Let (X, Σ, μ) be the Lebesgue measure space on $X = \mathbf{R}$. Modify the tree from Example 1 by taking $n \in \mathbf{Z}$ and $k \in \mathbf{Z}$ and let $X_{(n,k)}$ be formally as in Example 1. In these two examples, each $X_{(n,k)}$ has two children. This example is of Type III.

Example 3. Let (X, Σ, μ) be a weighted counting measure on $X = \mathbf{N}$ with $0 < \mu(n) < \infty$ for each $n \in \mathbf{N}$. Modify the tree from Example 1 by taking integers $n \leqslant 0$ and $k \in \mathbf{N}$. Let $X_{(n,k)} = X \cap (\, 2^{-n}\,(k - 1)\;,\; 2^{-n}\,(k)]$. Each $X_{(0,k)} = \{k\}$ is a leaf. This example is of Type II if $\mu(X) < \infty$ and of Type III if $\mu(X) \;=\; \infty$.

Example 4 (Logarithmic tree)**.** Let $M = \{1, 2, \ldots, m\}$ for some $m \in \mathbf{N}$. The tree $\mathcal{T}_{\text{log}}$ on M is uniquely determined by the following properties.

1. It has l generations $(0, \cdots, l-1)$ where $2^{l-2} < m \leqslant 2^{l-1}$.
2. Each element of $\mathcal{T}_{\text{log}}$ is a set of consecutive integers from M.
3. It has one root element $\rho = M$ and $g(\rho) = 0$.
4. The $(l-1)$ generation consists of the leaves $\{\{1\}, \{2\}, \ldots \{m\}\}$.
5. Each element of $\mathcal{T}_{\text{log}}$ with cardinality greater than 1 has two children and the cardinality of the youngest child is equal to or one less than the cardinality of the older child.

This tree will be used in the general wavelet construction. The name *logarithmic* comes from the fact that the number of generations behaves as the logarithm of $\#M$.

Example 5 (Linear tree)**.** Let M be as in the previous example. The tree $\mathcal{T}_{\text{lin}}$ on M is uniquely determined by the following properties.

1. It has m generations $(0, \cdots, m-1)$.
2. Each element of $\mathcal{T}_{\text{lin}}$ is a set of consecutive integers of M.
3. It has one root element $\rho = M$ and $g(\rho) = 0$.
4. The $(m-1)$ generation consists of the leaves $\{\{1\}, \{2\}, \ldots \{m\}\}$.
5. Each element of $\mathcal{T}_{\text{lin}}$ with cardinality greater than 1 has two children and the cardinality of the youngest child is 1.

This example will also be used in the general wavelet construction. It is called *linear* since the number of generations is proportional to the number of elements of M.

The previous two trees may be viewed as nested partitionings themselves.

Example 6. Let $(X = M, \Sigma, \mu)$ be a weighted counting measure on $M = \{1, 2, \ldots, m\}$ with $0 < \mu(n) < \infty$ for each $n \in M$. Each of the previous two examples gives a nested partitioning $\{M_\alpha \mid \alpha \in M\}$ of M by letting $M_\alpha = \alpha$.

Example 7. Let (X, Σ, μ) be the Lebesgue measure space on $X = \mathbf{R}^n$ for some $n \in \mathbf{N}$. Let $\{X_\alpha\}$ be the dyadic cubes in X. Each X_α has 2^n children. This example is of Type III.

Example 8. Let X be the sphere S^2 in $\mathbf{R}^3$, endowed with the surface area measure. Consider the icosahedron Π centered at the origin along with the corresponding map

$$P : \Pi \to \mathrm{S}^2 \qquad \text{where} \qquad P(v) = v / \|v\|_{\mathbf{R}^3} \ .$$

We use P to push a partition of Π out to a partition of S^2. The 0^{th}-generation partition consists of just $X_\rho = S^2$. Next obtain nested partitions of Π by recursively subdividing each triangular side. Figure 1 depicts a typical subdivision of a triangle. The image under P of these nested partitions of Π are nested partitions of S^2, where each set of a partition is a spherical triangle. Figure 2 shows the icosahedron (left), the icosahedron after 3 subdivisions (middle), and the result after applying P to the middle polyhedron

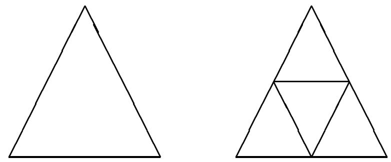

FIGURE 1. Subdividing a triangle.

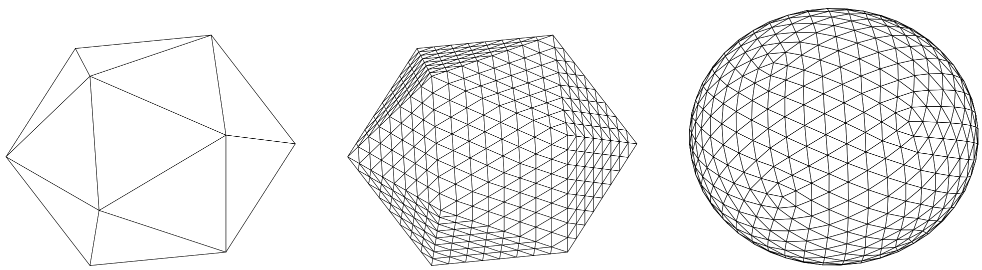

FIGURE 2. Partitioning a sphere.

(right). The latter is the fourth generation of the partitions on the sphere. This example is used in [20] as a starting point.

One can now consider a countable collection of disjoint measure spaces, each of which has a nested partitioning with respect to a forest. It is possible to unite their forests into a new forest. Then the union of their nested partitionings forms a nested partitioning for the disjoint sum of the measure spaces (with respect to the new forest). In this fashion, it is possible to combine the above examples.

Any measure space that has a nested partitioning is necessarily complete, separable, and σ-finite.

Fact 9. Each complete, separable, σ-finite measure space has a nested partitioning.

To see this, note that a complete separable σ-finite measure space may be viewed as a disjoint sum of complete separable measure spaces, with one space being purely atomic and the other spaces being purely non-atomic and of finite measure. As in Examples 3 and 6, one can build a nested partitioning on the purely atomic space. On each of the purely non-atomic spaces of finite measure, using Example 1 and a theorem of Carathéodory (cf. [23, I.B.1]), one can build a nested partitioning (with care, separability guarantees (P5)). Then, as noted above, these partitionings combine to give a nested partitioning for the entire space.

5. CONSTRUCTION OF WAVELETS

Plant a tree $\mathcal{T}$. Let $\{X_\alpha \mid \alpha \in \mathcal{T}\}$ be a nested partitioning for X with respect to $\mathcal{T}$. We are now ready to build on X our wavelets, which have as their basic building blocks the scaling functions $\{\varphi_\alpha\}_{\alpha\in\mathcal{T}}$

where

$$\varphi_\alpha = \mu(X_\alpha)^{-1/p}\, 1_{X_\alpha} \; .$$

The wavelets will be indexed by a set $\mathcal{G}$. The set $\mathcal{G}$ consists of a set $\mathcal{G}^*$ along with possibly one other element. When helpful, we will try to be consistent in the notation by denoting a parent by α, a child by β, and an element of $\mathcal{G}$ by γ.

First we concentrate on $\mathcal{G}^*$. Each wavelet indexed by a $\gamma \in \mathcal{G}^*$ will be of the form

$$\psi_\gamma = n_\gamma \left(\frac{1_{P_\gamma}}{\mu(P_\gamma)} - \frac{1_{N_\gamma}}{\mu(N_\gamma)} \right) \; , \tag{3}$$

for some sets P_γ and N_γ in $\widetilde{\Sigma}$ with n_γ chosen as to normalize ψ_γ in L_p, thus,

$$n_\gamma = \left(\mu(P_\gamma)^{1-p} + \mu(N_\gamma)^{1-p} \right)^{-1/p} \; .$$

This resembles the definition of a Haar wavelet, but as $\mu(P_\gamma)$ can differ from $\mu(N_\gamma)$, we refer to it as an *unbalanced Haar wavelet*. It is constructed to have zero mean.

The set $\mathcal{G}^*$ has the form

$$\mathcal{G}^* = \bigcup_{\alpha \in \mathcal{T}} G(\alpha) \; ,$$

where the set $G(\alpha)$ contains $\max(0, m-1)$ elements ($m = \#C(\alpha)$) and is constructed as follows. The basic idea is to use $G(\alpha)$ to index those unbalanced Haar wavelets that will be supported on X_α and constant on X_β where $\beta \in C(\alpha)$. To do this, we build a mini-tree amongst the children. Enumerate the children of α as β_i with $i \in M = \{1, 2, \cdots, m\}$ and $\beta_i < \beta_{i+1}$. Next consider a tree $\mathcal{T}_M$ that is either $\mathcal{T}_{\mathrm{lin}}$ or $\mathcal{T}_{\mathrm{log}}$. Let

$$G(\alpha) = \{ (\alpha, \zeta) \in \{\alpha\} \times \mathcal{T}_M \mid \#C(\zeta) = 2 \} \; .$$

Note that each element of $\mathcal{T}_M$ has at most two children.

The element $\gamma = (\alpha, \zeta) \in G(\alpha)$ generates a wavelet ψ_γ as in (3) with

$$P_\gamma = \bigcup_{i \in \zeta_1} X_{\beta_i}, \quad \text{and} \quad N_\gamma = \bigcup_{i \in \zeta_2} X_{\beta_i} \; ,$$

where ζ_1 and ζ_2 are the two children of ζ.

The remainder of $\mathcal{G}$ depends on the particular type of splitting.

- For Type I, let

$$\mathcal{G} = \mathcal{G}^* \cup \{\rho\} \quad \text{and} \quad \psi_\rho = \varphi_\rho \;\equiv\; \mu(X)^{-1/p}\, 1_X \; .$$

  For later use, let $P_\rho = X$ and $N_\rho = \emptyset$.

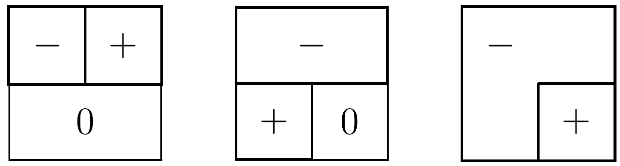

FIGURE 3. Mitrea wavelets (linear construction).

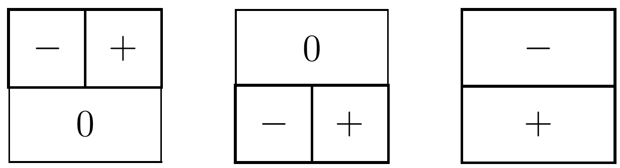

FIGURE 4. Logarithmic construction.

- For Type II, let

$$\mathcal{G} = \mathcal{G}^* \cup \{\nu\} \quad \text{and} \quad \psi_\nu = \mu(X)^{-1/p}\, 1_X \ ,$$

  where $\nu \notin \mathcal{G}^*$. For later use, let $P_\nu = X$ and $N_\nu = \emptyset$.
- For Type III, let $\mathcal{G} = \mathcal{G}^*$.

Finally, take

$$\Psi = \{\psi_\gamma \mid \gamma \in \mathcal{G}\} \ . \tag{4}$$

We define the following partial ordering on $\mathcal{G}$. Any element in $\mathcal{G}^*$ is less than an element in $\mathcal{G} \setminus \mathcal{G}^*$. Next consider two elements in $\mathcal{G}^*$, say $\gamma_1 = (\alpha_1, \zeta_1)$ and $\gamma_2 = (\alpha_2, \zeta_2)$. Now $\gamma_1 < \gamma_2$ if either $\alpha_1 < \alpha_2$, or $\alpha_1 = \alpha_2$ and $\zeta_1 < \zeta_2$. Also define the generation function on $\mathcal{G}^*$ as $g((\alpha, \zeta)) = g(\alpha)$.

Depending on the choices $\mathcal{T}_{\text{lin}}$ or $\mathcal{T}_{\text{log}}$ for $\mathcal{T}_M$ we refer to the respective construction as linear or logarithmic. They will have the same theoretical properties; the advantage of the logarithmic construction is that the support size of the wavelets is smaller and that the wavelets are have more symmetry. In the setting of the examples in the previous section, the above construction leads to well-known wavelets.

- For Example 1 (resp. 2), Ψ is the Haar system on $[0, 1)$ (resp. $\mathbf{R}$).
- For Example 3 (resp. 6), Ψ is a Haar-like unconditional basis for ℓ_p (resp. ℓ_p^m).
- For Example 7 the linear construction leads to the Mitrea wavelets [1, 15]. The Mitrea wavelets are the first example of higher dimensional compactly supported Haar-like wavelets in the case of non-translation invariant measures. The basic idea, as depicted in Figure 3 for the two dimensional case, is simple but extremely clever. Our linear construction is inspired by the Mitrea wavelets. The setting of the Mitrea wavelets is actually more general than presented here as the measure can be Clifford-algebra-valued.
- Again for Example 7, Figure 4 depicts the logarithmic construction in case $n = 2$.

Concluding, the basic idea behind constructing the unbalanced Haar wavelets in the case that the number of children is greater than two is to build a mini-tree amongst the children as to reduce it to the case of two children.

6. Properties of wavelets

Clearly Ψ is normalized. Note that for each $\gamma \in \mathcal{G}^*$,

$$\int_X \psi_\gamma \, d\mu = 0 \, . \tag{5}$$

If γ and γ' are in $\mathcal{G}$, then

$$\int_X \psi_\gamma \, \psi_{\gamma'} \, d\mu = 0 \, , \tag{6}$$

for if ψ_γ and $\psi_{\gamma'}$ are not disjointly supported and $\gamma < \gamma'$, then $\psi_{\gamma'}$ is constant on the support of ψ_γ. If $\alpha \in \mathcal{T}$, then

$$\operatorname{span}\{\varphi_\beta \mid \beta \in C_l(\alpha)\} \; = \; \operatorname{span}\{\varphi_\alpha, \, \psi_\gamma \mid \gamma \in G(\alpha)\} \, , \tag{7}$$

and furthermore this extends over several generations for if $i \in \mathbf{N}$, then

$$\operatorname{span}\{\varphi_\beta \mid \beta \in C_l^i(\alpha)\} = \operatorname{span}\left\{\varphi_\alpha, \, \psi_\gamma \mid \gamma \in \cup_{j=0}^{i-1} G(C^j(\alpha))\right\} \, . \tag{8}$$

To see this, note that set containment in one direction ($\supseteq$) is clear; furthermore, in the right-hand side (by (5) and (6)) and the left-hand side, the indicated functions that span the space (of dimension $\#C_l^i(\alpha)$) can be viewed as an orthogonal basis.

Since

$$\mathrm{L}_p = \operatorname{clos}\operatorname{span}\{1_A \mid A \in \Sigma^+\} \, ,$$

and $\{X_\alpha \mid \alpha \in \mathcal{T}\} \cup \emptyset$ is a semi-ring that generates $\widetilde{\Sigma}$, it follows that (cf. [3, Theorem 11.4])

$$\mathrm{L}_p = \operatorname{clos}\operatorname{span}\Phi \tag{9}$$

where

$$\Phi = \{\varphi_\alpha \mid \alpha \in \mathcal{T}\} \, .$$

Note that L_p is separable since $\mathcal{T}$ is countable.

Lemma 10.

$$\operatorname{clos}\operatorname{span}\Psi = \mathrm{L}_p.$$

Proof. Fix an indicator function 1_{X_δ} with $\delta \in \mathcal{T}$. By (9), it suffices to show that

$$1_{X_\delta} \in \operatorname{clos}\operatorname{span}\Psi \ . \tag{10}$$

If $\delta = \rho$, then (10) clearly holds; thus, assume that $\delta \neq \rho$.

Consider $\alpha \in \mathcal{T}$ of the form $\alpha = p^i(\delta)$ for some $i > 0$, along with the corresponding function

$$f_i = 1_{X_\delta} - \frac{\mu(X_\delta)}{\mu(X_\alpha)} 1_{X_\alpha} \ .$$

Towards showing that f_i is in $\operatorname{span}\Psi$, consider (8) with the same α and i as in this proof. Clearly, f_i is in the set on the left-hand side and so it is also in the set on the right-hand side. Since f_i and ψ_γ have a vanishing integral (for $\gamma \in \mathcal{G}^*$) while φ_α does not, f_i does not have a component along φ_α. Thus

$$f_i \in \operatorname{span}\left\{\psi_\gamma \mid \gamma \in \cup_{j=0}^{i-1} G(C^j(\alpha))\right\} \ \subset \ \operatorname{span}\Psi \ .$$

Towards (10), consider the three types of nested partitionings separately.

- For Type I, $p^i(\delta) = \rho$ for some finite i and $1_{X_\rho} \in \operatorname{span}\Psi$.
- For Type II, the function $1_{X_{p^i(\delta)}}$ tends to 1_X in L_p-norm as $i \to \infty$ and $1_X \in \operatorname{span}\Psi$.
- For Type III,

$$\|f_i - 1_{X_\delta}\|_p = \mu(X_\delta)\, \mu(X_{p^i(\delta)})^{-1/q} \ ,$$

  and $\mu(X_{p^i(\delta)})$ tend to infinity as $i \to \infty$.

So for each of the three types, (10) holds. $\square$

Thus Ψ satisfies (C1). Toward (C2), recall that a constant K_p exists so that for the usual Haar functions $\{h_k\}_{k=1}^\infty$ on $[0,1)$,

$$\left\|\sum_{k=1}^n \epsilon_k\, c_k\, h_k\right\|_p \leqslant K_p \left\|\sum_{k=1}^n c_k\, h_k\right\|_p \ ,$$

for all $n \in \mathbf{N}$, sequences $\{c_k\}_{k=1}^n$ in $\mathbf{R}$, and $\epsilon_k = \pm 1$. This inequality (in an equivalent formulation) is due to R.E.A.C. Paley [18]; the above formulation was noted by Marcinkiewicz [13]. Using martingale theory, Burkholder [5] generalized Paley's inequality to hold for martingale difference sequences (such as the Haar functions) on $[0,1)$. Since the wavelets Ψ can essentially be viewed as a martingale difference sequence, we will call upon Burkholder's generalization. We first recall some basic definitions.

Fix $X_0 \in \Sigma^+$ and a sub-σ-field Σ_0 of Σ that is generated by a partition $\pi = \{E_1, \ldots, E_n\}$ of X_0 (thus $X \setminus X_0$ is an atom of Σ_0). Consider $g \in \mathrm{L}_1(X, \Sigma, \mu)$ with $\operatorname{supp} g \subset X_0$. Then the conditional expectation $E(g \mid \Sigma_0)$ of g with respect to Σ_0 is

$$E(g \mid \Sigma_0) \ = \ \sum_{i=1}^n \frac{\int_{E_i} g\, d\mu}{\mu(E_i)}\, 1_{E_i} \ ,$$

observing the convention that $0/0$ is 0. A *simple martingale*, with respect to a non-decreasing sequence $\{\Sigma_i\}_{i=1}^n$ of sub-σ-fields of Σ, is a finite sequence $\{f_i\}_{i=1}^n$ of simple functions with finite support that satisfy that f_i is Σ_i-measurable for $1 \leqslant i \leqslant n$ and that $E(f_{i+1} \mid \Sigma_i) = f_i$ for $1 \leqslant i < n$. Its corresponding *martingale difference sequence* $\{d_i\}_{i=1}^n$ is given by $d_i = f_i - f_{i-1}$ where f_0 is just the null function, thus $f_k = \sum_{i=1}^k d_i$.

Our setting calls for the following version of Burkholder's celebrated inequality.

Theorem 11 (Burkholder)**.** *If $\{f_i\}_{i=1}^n$ is a simple martingale with respect to a non-decreasing sequence $\{\Sigma_i\}_{i=1}^n$ of sub-σ-fields of Σ, then its corresponding martingale difference sequence $\{d_i\}_{i=1}^n$ satisfies*

$$\left\| \sum_{i=1}^n \epsilon_i \, c_i \, d_i \right\|_p \leqslant (p^* - 1) \left\| \sum_{i=1}^n c_i \, d_i \right\|_p , \tag{11}$$

for all $n \in \mathbf{N}$ and all choices of $c_i \in \mathbf{R}$ and $\epsilon_i = \pm 1$.

See [5, 6, 7] for the proof.

If $\mathrm{L}_p(m)$ isometrically embeds into $\mathrm{L}_p(X, \Sigma, \mu)$, then any basis for $\mathrm{L}_p(X, \Sigma, \mu)$ has, for each $\epsilon > 0$, a blocked basis that is $(1+\epsilon)$-equivalent to the usual Haar basis ([16, 17] and [11]). Burkholder [4] showed that the unconditional basis constant of the usual Haar basis is $p^* - 1$. Olevskiĭ showed that the unconditional basis constant of any unconditional basis is greater than or equal to that of the Haar system. From these facts follow the below known fact.

Theorem 12. *If (X, Σ, μ) is not purely atomic, then the unconditional basis constant for any unconditional basis for $\mathrm{L}_p(X, \Sigma, \mu)$ is at least $(p^* - 1)$.*

The following lemma is needed to apply Theorem 11 to finite subsets of Ψ.

Lemma 13. *Fix a finite subset $\{\gamma_i\}_{i=1}^n$ from $\mathcal{G}$ that satisfies $\gamma_1 > \gamma_2 > \ldots > \gamma_n$. Let $X_0 \in \Sigma^+$ be such that $\sup \psi_{\gamma_i} \subset X_0$ for each i. Consider the corresponding partitions*

$$\pi_i = \{P_{\gamma_i} \, , \, N_{\gamma_i} \, , \, X_0 \setminus (P_{\gamma_i} \cup N_{\gamma_i})\}$$

of X_0 and let $\Sigma_i = \sigma\left(\{\pi_j \mid 1 \leqslant j \leqslant i\}\right)$. Then

1. *ψ_{γ_i} is Σ_i-measurable for $i = 1, \ldots, n$*
2. *$E(\psi_{\gamma_{i+1}} \mid \Sigma_i) = 0$ for $i = 1, \ldots, n-1$.*

Proof. Since ψ_{γ_i} is constant on each of the sets P_{γ_i}, N_{γ_i}, and $X_0 \setminus (P_{\gamma_i} \cup N_{\gamma_i})$, it is Σ_i-measurable. Fix $1 \leqslant i < n$ and consider $E(\psi_{\gamma_{i+1}} \mid \Sigma_i)$. An atom $A \subset X_0$ of Σ_i has the form

$$A = \bigcap_{k=1}^m F_k$$

where

$$F_k \in \bigcup_{j=1}^i \{P_{\gamma_j} \, , \, N_{\gamma_j} \, , \, X_0 \setminus (P_{\gamma_j} \cup N_{\gamma_j})\}$$

and $\gamma_{i+1} < \gamma_j$ for $j = 1, \dots, i$. If $\gamma_{i+1} < \gamma \in \mathcal{G}$, then P_γ (and likewise for N_γ and for $X_0 \setminus (P_\gamma \cup N_\gamma)$) is either disjoint from or contains the support of $\psi_{\gamma_{i+1}}$. Furthermore, $\int_X \psi_{\gamma_{i+1}}\, d\mu$ is zero. Thus $E(\psi_{\gamma_{i+1}} \mid \Sigma_i) = 0$, as needed. □

Now plant a whole forest $\mathcal{F}$. Let $\{X_\alpha \mid \alpha \in \mathcal{F}\}$ be a nested partitioning for X with respect to $\mathcal{F}$. Keeping with the notation from Section 4, write X as the disjoint union of $X(\varkappa)$'s. For each $\varkappa \in \mathcal{K}$, the subcollection $\{X_\alpha \mid \alpha \in \mathcal{F}(\varkappa)\}$ is a nested partitioning for $X(\varkappa)$ with respect to the tree $\mathcal{F}(\varkappa)$; thus, there are wavelets $\Psi(\varkappa)$ as in (4) on $X(\varkappa)$. Let

$$\Xi = \bigcup_{\varkappa \in \mathcal{K}} \Psi(\varkappa) \,.$$

We are now able to state the main result of this paper.

Corollary 14. *The wavelets* Ξ *forms a normalized unconditional basis for* $\mathrm{L}_p(X, \Sigma, \mu)$, *with* $K_p(\Xi) \leqslant (p^* - 1)$. *If* L_p *is not purely atomic, then* $K_p(\Xi) = (p^* - 1)$.

Proof. In light of Lemma 10 and Theorem 12, it suffices to show that (C2) holds with $K = p^* - 1$ for the set Ξ. Since X is the *disjoint* union of the $X(\varkappa)$'s, for any $f \in \mathrm{L}_p$

$$\|f\|_p^p \;=\; \sum_{\varkappa \in \mathcal{K}} \left\| f\, 1_{X(\varkappa)} \right\|_p^p \,.$$

Furthermore, for each $\varkappa \in \mathcal{K}$ and $\gamma \in \mathcal{G}$

$$\psi_\gamma\, 1_{X(\varkappa)} \;=\; \begin{cases} \psi_\gamma & \text{if } \gamma \in \mathcal{F}(\varkappa) \\ 0 & \text{if } \gamma \notin \mathcal{F}(\varkappa) \,. \end{cases}$$

Thus it suffices to show that (C2) holds with $K = p^* - 1$ for each set $\Psi(\varkappa)$. Thus we assume, without loss of generality, that $\mathcal{F}$ is a tree and denote Ξ by just Ψ.

Keeping with previous notation, fix a finite collection $\Gamma \subset \mathcal{G}$ and order $\Gamma = \{\gamma_i\}_{i=1}^n$ so that $\gamma_1 > \gamma_2 > \ldots > \gamma_n$. Let $\{\pi_i\}_{i=1}^n$ and $\{\Sigma_i\}_{i=1}^n$ be as in the statement of Lemma 13. By Theorem 11, it suffices to show that the sequence $\{f_i\}_{i=1}^n$ given by

$$f_i \;=\; \sum_{j=1}^{i} \psi_{\gamma_j}$$

is a simple martingale with respect to $\{\Sigma_i\}_{i=1}^n$.

Lemma 13 gives that each f_i is Σ_i-measurable. If $i < n$, then by Lemma 13 and the linearity of the conditional expectation operator

$$E(f_{i+1} \mid \Sigma_i) - E(f_i \mid \Sigma_i) = E(f_{i+1} - f_i \mid \Sigma_i) = E(\psi_{\gamma_{i+1}} \mid \Sigma_i) = 0 \,.$$

Since f_i is Σ_i-measurable, $f_i = E(f_i \mid \Sigma_i)$. Thus $E(f_{i+1} \mid \Sigma_i) = f_i$, as needed. □

Since Ξ is an unconditional basis for L_p, each ordering $\{\psi_{\gamma_i}\}_{i=1}^{\infty}$ of Ξ forms a basis for L_p. Given an ordering $\{\gamma_i\}$, the basis constant $M_p(\{\psi_{\gamma_i}\})$ is the smallest number M for which

$$\left\| \sum_{i=1}^{n} c_i \, \psi_{\gamma_i} \right\|_p \leqslant M \left\| \sum_{i=1}^{m} c_i \, \psi_{\gamma_i} \right\|_p$$

holds for all $n, m \in \mathbf{N}$ with $n < m$ and all choices of $c_i \in \mathbf{R}$. Clearly, $1 \leqslant M_p(\{\psi_{\gamma_i}\}) \leqslant K_p(\Xi)$. In fact, if $P_p(\{\Xi\})$ is the supremum of $M_p(\{\psi_{\gamma_i}\})$ over all possible orderings of Ξ, then $P_p(\{\Xi\}) \leqslant K_p(\{\Xi\}) \leqslant 2P_p(\{\Xi\})$. If $M_p(\{\psi_{\gamma_i}\}) = 1$, then $\{\psi_{\gamma_i}\}$ is a monotone basis.

A rooted tree can be enumerate $\{\gamma_i\}_i$ so that $\gamma_i > \gamma_{i+1}$.

Corollary 15. *The wavelets $\Psi = \{\psi_{\gamma_i}\}$ associated with a rooted tree form a monotone basis for L_p when ordered so that $\gamma_i > \gamma_{i+1}$.*

Proof. Fix $n \in \mathbf{N}$ and a sequence $\{c_i\}_{i=1}^{n+1}$ from $\mathbf{R}$. By Corollary 14, it suffices to show that

$$\left\| \sum_{i=1}^{n} c_i \, \psi_{\gamma_i} \right\|_p \leqslant \left\| \sum_{i=1}^{n+1} c_i \, \psi_{\gamma_i} \right\|_p .$$

Consider the sub-σ-field $\Sigma_n = \sigma\left(\{\pi_j \mid 1 \leqslant j \leqslant n\}\right)$ as in the statement of Lemma 13 with $X_0 = X$. It follows from Lemma 13 that $\sum_{i=1}^{n} c_i \, \psi_{\gamma_i}$ is Σ_n-measurable and that $E(\psi_{\gamma_{n+1}} \mid \Sigma_n) = 0$. Thus

$$\begin{aligned} \sum_{i=1}^{n} c_i \, \psi_{\gamma_i} &= E\left(\sum_{i=1}^{n} c_i \, \psi_{\gamma_i} \mid \Sigma_n \right) \\ &= E\left(\sum_{i=1}^{n} c_i \, \psi_{\gamma_i} \mid \Sigma_n \right) + c_{n+1} E\left(\psi_{\gamma_{n+1}} \mid \Sigma_n \right) \\ &= E\left(\sum_{i=1}^{n+1} c_i \, \psi_{\gamma_i} \mid \Sigma_n \right) . \end{aligned}$$

The result now follows from the fact that conditional expectation is a contraction on L_p. ☐

7. Dual basis and characterization

Consider the coordinate functionals $\{\widetilde{\psi}_\gamma \mid \gamma \in \mathcal{G}\}$ of the unconditional basis Ξ, which are (uniquely) determined by the condition $\langle \psi_{\gamma'}, \widetilde{\psi}_\gamma \rangle = \delta_{\gamma\gamma'}$. Since L_p is reflexive, $\{\widetilde{\psi}_\gamma \mid \gamma \in \mathcal{G}\}$ forms an unconditional basis for the dual space $\mathrm{L}_q(X, \Sigma, \mu)$. Thus, if $f \in \mathrm{L}_p$ and $g \in \mathrm{L}_q$, then

$$f = \sum_{\gamma \in \mathcal{G}} \langle f, \widetilde{\psi}_\gamma \rangle \, \psi_\gamma \quad \text{and} \quad g = \sum_{\gamma \in \mathcal{G}} \langle \psi_\gamma, g \rangle \, \widetilde{\psi}_\gamma \, , \tag{12}$$

where the convergence is unconditional.

It follows from (6) that $\widetilde{\psi}_\gamma$ is a multiple of ψ_γ. Straightforward calculations give that if $\gamma \in \mathcal{G}^*$ then

$$\widetilde{\psi}_\gamma = \widetilde{n}_\gamma \left(\frac{1_{P_\gamma}}{\mu(P_\gamma)} - \frac{1_{N_\gamma}}{\mu(N_\gamma)} \right) \, ,$$

where

$$\widetilde{n}_\gamma = \left[n_\gamma (\mu(P_\gamma)^{-1} + \mu(N_\gamma)^{-1}) \right]^{-1} ,$$

and if $\rho \in \mathcal{G}$ then

$$\widetilde{\psi}_\rho = \mu(X)^{-1/q} \, 1_X ,$$

while if $\nu \in \mathcal{G}$ then

$$\widetilde{\psi}_\nu = \mu(X)^{-1/q} \, 1_X .$$

It follows from Corollary 14 that $1 \leqslant \left\| \widetilde{\psi}_\gamma \right\|_q \leqslant 2(p^* - 1)$ for each $\gamma \in \mathcal{G}$. If $\gamma \in \mathcal{G} \setminus \mathcal{G}^*$, then $\left\| \widetilde{\psi}_\gamma \right\|_q = 1$. For a fixed $\gamma \in \mathcal{G}^*$,

$$\left\| \widetilde{\psi}_\gamma \right\|_q = \widetilde{n}_\gamma \left(\mu(P_\gamma)^{1-q} + \mu(N_\gamma)^{1-q} \right)^{1/q} ,$$

which need not be one. We examine this a little closer. Let

$$r = \mu(P_\gamma)/\mu(N_\gamma) \quad \text{and} \quad N(p,r) = \left\| \widetilde{\psi}_\gamma \right\|_q .$$

Then

$$N(p,r) = (1 + r^{1-p})^{1/p} \, (1 + r^{1-q})^{1/q} \, (1 + r^{-1})^{-1}.$$

The following properties of $N(p,r)$ hold for $1 < p < \infty$ and $0 < r < \infty$:

1. $N(p,r) = N(q,r)$,
2. $N(p,r) = N(p,1/r)$,
3. $N(p,1) = N(2,r) = 1$,
4. $1 \leqslant N(p,r) \leqslant 2$,
5. for any fixed p, $\lim_{r\to\infty} N(p,r) = 1$,
6. for any fixed r, $\lim_{p\to\infty} N(p,r) = 2\,(1 + r^{-1})^{-1}$.

These easily can be verified. The uniform bound in (4) follows from bounding each factor which yields $2^{1/p}\, 2^{1/q}\, 1$. Thus the norm of the coordinate functional is always less than 2, while the last property shows that it can be arbitrarily close to 2. If $p = 2$ or if $\mu(P_\gamma) = \mu(N_\gamma)$ for each $\gamma \in \mathcal{G}^*$, then the dual basis is normalized.

Following the reasoning in [14, Chapter 6], we now derive a criterion, connected with the absolute value of the coefficients c_γ, to determine whether a formal series $\sum c_\gamma \, \psi_\gamma$ belongs to L_p. Towards this, we consider the Cantor group $\Delta \equiv \{-1,1\}^{\mathcal{G}}$ of all sequences ϵ (indexed by $\mathcal{G}$) of ± 1, along with its coordinate functionals $e_\gamma : \Delta \to \{-1,1\}$ determined by $\epsilon = \{e_\gamma(\epsilon)\}_{\gamma \in \mathcal{G}}$, and its product (i.e. Bernoulli probability) measure ζ.

For each ϵ in Δ, let $T_\epsilon : \mathrm{L}_p \to \mathrm{L}_p$ be the continuous (by (C2)) linear operator determined by

$$T_\epsilon \, \psi_\gamma = e_\gamma(\epsilon) \, \psi_\gamma .$$

Consider a function $f \in \mathrm{L}_p$ of the form

$$f = \sum_{\gamma\in\Gamma} c_\gamma \, \psi_\gamma \tag{13}$$

for some finite subset Γ of $\mathcal{G}$. So if $\epsilon \in \Delta$ and $x \in X$ then

$$(T_\epsilon f)(x) = \sum_{\gamma\in\Gamma} e_\gamma(\epsilon)\, c_\gamma \, \psi_\gamma(x) \ .$$

It follows from (C2) that

$$K_p^{-p} \ \|f\|_p^p \leqslant \|T_\epsilon \, f\|_p^p \leqslant K_p^p \ \|f\|_p^p \tag{14}$$

where K_p is the unconditional basis constant of Ξ.

Since $T_\epsilon f(x)$ is product $(\mu \times \zeta)$-measurable, Tonelli's Theorem gives that

$$\int_\Delta \|T_\epsilon f\|_p^p \ d\zeta(\epsilon) \ = \ \int_X \int_\Delta \left| \sum_{\gamma\in\Gamma} e_\gamma(\epsilon)\, c_\gamma \, \psi_\gamma(x) \right|^p \ d\zeta(\epsilon)\, d\mu(x) \ . \tag{15}$$

Khinchin's inequality (cf. [24]) gives strictly positive constants c_p and C_p so that

$$c_p \ \|\{a_\gamma\}\|_{\ell_2} \leqslant \left(\int_\Delta \left| \sum_{\gamma\in\Gamma} a_\gamma \, e_\gamma(\epsilon) \right|^p \ d\zeta(\epsilon) \right)^{1/p} \leqslant C_p \ \|\{a_\gamma\}\|_{\ell_2} \tag{16}$$

for each sequence $\{a_\gamma\}_{\gamma\in\Gamma}$ of real numbers. Combining (16) and (15) yields

$$c_p^p \int_X |(Af)(x)|^p \ d\mu(x) \leqslant \int_\Delta \|T_\epsilon f\|_p^p \ d\zeta(\epsilon) \leqslant C_p^p \int_X |(Af)(x)|^p \ d\mu(x) \tag{17}$$

where

$$(A\, f)(x) = \left(\sum_{\gamma\in\Gamma} |c_\gamma|^2 \ |\psi_\gamma(x)|^2 \right)^{1/2} \ . \tag{18}$$

Next, integrate inequality (14) over Δ, note that $\zeta(\Delta) = 1$, and use (17) to see that

$$c_p \, K_p^{-1} \ \|A\, f\|_p \leqslant \|f\|_p \leqslant K_p \, C_p \ \|A\, f\|_p \ . \tag{19}$$

Consider any ordering $\{\psi_{\gamma_i}\}_{i=1}^\infty$ of Ξ and a function $f \in \mathrm{L}_p$. The functions

$$f_n = \sum_{i=1}^n c_{\gamma_i} \, \psi_{\gamma_i} \quad \text{where} \quad c_{\gamma_i} \ = \ \langle \, f, \widetilde{\psi}_{\gamma_i} \, \rangle$$

are of the form in (13) and thus satisfy (19). Furthermore, $\{f_n\}$ converges in L_p-norm to f and $\{(Af_n)\}$ is a μ-a.e. increasing sequence of non-negative L_p-functions. Thus the (non-linear) mapping A in (18) extends to a mapping from L_p to L_p. Now follows the below characterization.

Theorem 16. *If $1 < p < \infty$, then*

$$\sum_{\gamma \in \mathcal{G}} c_\gamma \, \psi_\gamma \in \mathrm{L}_p \qquad \Longleftrightarrow \qquad \left(\sum_{\gamma \in \mathcal{G}} |c_\gamma|^2 \; |\psi_\gamma(x)|^2 \right)^{1/2} \in \mathrm{L}_p \, .$$

8. Multiresolution analysis

Wavelets are closely related to the concept of multiresolution analysis [10, 12, 14]. Traditionally wavelets are the translates and dilates of one particular function. Since we work with arbitrary partitions and non-translation invariant measures, our wavelets cannot be the translates and dilates of one function. In fact, they are a special case of so-called "second generation wavelets". The basic idea of second generation wavelets is to give up the translation and dilation structure of wavelets, but to keep their desirable properties such as multiresolution analysis and fast transform algorithms. In this section we show how the unbalanced Haar wavelets fit into this concept. The fast wavelet transform will give us an algorithm to compute the coefficients in the expansions (12).

We define two new sets as

$$S(\gamma) = C_l(\alpha) \quad \text{if} \quad \gamma \in G(\alpha) \quad \text{and} \quad S^*(\beta) = G(\alpha) \quad \text{if} \quad \beta \in C_l(\alpha) \, .$$

Now consider (7). The basis $\{\varphi_\beta \mid \beta \in C_l(\alpha)\}$ has dual basis

$$\{\widetilde{\varphi}_\beta \mid \beta \in C_l(\alpha)\} \, ,$$

where $\widetilde{\varphi}_\beta$ is a multiple of φ_β and $\|\widetilde{\varphi}_\beta\|_q = 1$, while the other basis $\{\varphi_\alpha, \, \psi_\gamma \mid \gamma \in G(\alpha)\}$ has dual basis

$$\{\widetilde{\varphi}_\alpha, \, \widetilde{\psi}_\gamma \mid \gamma \in G(\alpha)\} \, ,$$

where $\widetilde{\psi}_\gamma$ is as in Section 7. The basis functions in the above two bases are related as follows:

(R1) for $\alpha \in \mathcal{F}$ and $\gamma \in \mathcal{G}^*$,

$$\varphi_\alpha = \sum_{\beta \in C_l(\alpha)} h_{\alpha,\beta} \, \varphi_\beta \quad \text{and} \quad \psi_\gamma = \sum_{\beta \in S(\gamma)} g_{\gamma,\beta} \, \varphi_\beta$$

where

$$h_{\alpha,\beta} = \langle \, \varphi_\alpha, \widetilde{\varphi}_\beta \, \rangle \quad \text{and} \quad g_{\gamma,\beta} = \langle \, \psi_\gamma, \widetilde{\varphi}_\beta \, \rangle$$

(R2) for $\beta \neq \rho$,

$$\varphi_\beta = \widetilde{h}_{p(\beta),\beta} \, \varphi_{p(\beta)} + \sum_{\gamma \in S^*(\beta)} \widetilde{g}_{\gamma,\beta} \, \psi_\gamma$$

where

$$\widetilde{h}_{\alpha,\beta} = \langle \, \varphi_\beta, \widetilde{\varphi}_\alpha \, \rangle \quad \text{and} \quad \widetilde{g}_{\gamma,\beta} = \langle \, \varphi_\beta, \widetilde{\psi}_\gamma \, \rangle \, .$$

For $k \in g(\mathcal{F})$, let $\mathcal{G}_k = \{\gamma \in \mathcal{G}^* \mid g(\gamma) = k\}$ and consider the subspaces V_k and W_k of L_p, where

$$V_k = \operatorname{clos\,span}\{\varphi_\beta \mid \beta \in \mathcal{F}_k^*\} \quad \text{and} \quad W_k = \operatorname{clos\,span}\{\psi_\gamma \mid \gamma \in \mathcal{G}_k\}\ .$$

Note that the indicated functions not only span, but also provide an unconditional basis for, these subspaces. The dual basis for this basis of V_k is given by

$$\{\widetilde{\varphi}_\beta \mid \beta \in \mathcal{F}_k^*\}$$

while the dual basis for this basis of W_k is given by

$$\{\widetilde{\psi}_\gamma \mid \gamma \in \mathcal{G}_k\}\ .$$

By viewing $\mathcal{F}_{k-1}^* \cup \mathcal{F}_k^*$ as a two-generation forest, it follows that

$$V_k = V_{k-1} \oplus W_{k-1} \tag{20}$$

and that V_k has another basis

$$\{\varphi_\alpha \mid \alpha \in \mathcal{F}_{k-1}^*\} \cup \{\psi_\gamma \mid \gamma \in \mathcal{G}_{k-1}\}$$

with dual basis

$$\{\widetilde{\varphi}_\alpha \mid \alpha \in \mathcal{F}_{k-1}^*\} \cup \{\widetilde{\psi}_\gamma \mid \gamma \in \mathcal{G}_{k-1}\}\ .$$

A function $f \in V_k$ has a representation as

$$f = \sum_{\beta \in \mathcal{F}_k^*} a_\beta\, \varphi_\beta \quad \text{with} \quad a_\beta = \langle\, f, \widetilde{\varphi}_\beta \,\rangle \tag{21}$$

as well as, by (20),

$$f = \sum_{\alpha \in \mathcal{F}_{k-1}^*} a_\alpha\, \varphi_\alpha + \sum_{\gamma \in \mathcal{G}_{k-1}} c_\gamma\, \psi_\gamma \tag{22}$$

with

$$a_\alpha = \langle\, f, \widetilde{\varphi}_\alpha \,\rangle \quad \text{and} \quad c_\gamma = \langle\, f, \widetilde{\psi}_\gamma \,\rangle\ .$$

The relations between the different representations follow from simple linear algebra arguments. To simplify notation, assume that the forest has no leaves, indeed, just replace each $\mathcal{F}_k$ with $\mathcal{F}_k^*$. Combining (21) and (R2) and identifying coefficients results in

$$a_\alpha = \sum_{\beta \in C(\alpha)} \widetilde{h}_{\alpha,\beta}\, a_\beta \quad \text{and} \quad c_\gamma = \sum_{\beta \in S(\gamma)} \widetilde{g}_{\gamma,\beta}\, a_\beta\ , \tag{23}$$

where $g(\beta) = k$ and $g(\alpha) = g(\gamma) = k - 1$. Similarly, combining (22) and (R1) results in

$$a_\beta = h_{p(\beta),\beta}\, a_{p(\beta)} + \sum_{\gamma \in S^*(\beta)} g_{\gamma,\beta}\, c_\gamma\ . \tag{24}$$

Next consider a function $f \in V_n$ with n fixed. Given the scaling function coefficients a_β with $g(\beta) = n$, we can recursively use (23) to calculate all wavelet coefficients c_γ on the older generations where

$g(\gamma) < n$. Conversely, given the coefficients a_α with $g(\alpha) = m$ along with all the wavelet coefficients c_γ where $m \leqslant g(\gamma) < n$, we can recursively use (24) to find the coefficients a_β on the younger generation where $g(\beta) = n$.

These operations form the so-called "fast wavelet transform". Since all summations in the transform are finite, it can easily be implemented on a computer. One only needs to build a forest data structure that satisfies all the forest properties of Section 3. This can be done nicely using an object-oriented programming language.

The way the algorithm is described, the number of operations for the calculation of the wavelet coefficients c_γ with $\gamma \in G(\alpha)$ is $\mathcal{O}(m^2)$. Actually, by using the hierarchy of $\mathcal{T}_M$, one can reduce this to $\mathcal{O}(m)$. In Example 7 (dyadic cubes on $\mathbf{R}^n$) we have $m = 2^n$ and this difference can become important. We do not include the details of this algorithm as they are straightforward.

9. Euclidean spaces

One of the original motivations for the construction of the unbalanced Haar wavelets was their use in practical applications; thus, we take a closer look at the case where X is a subset of $\mathbf{R}^n$.

Consider the topology induced by the Euclidean distance d on $\mathbf{R}^n$. Let X be a Borel set of $\mathbf{R}^n$ and $\mathcal{B}(X)$ be the Borel subsets of X. Consider a σ-finite measure μ on $\mathcal{B}(X)$ and let $(X, \mathcal{M}(X), \mu)$ be the μ-completion of $(X, \mathcal{B}(X), \mu)$. A common practical example is a weighted measure μ where $d\mu = w\, dm$ for the Lebesgue measure m on X and a non-negative Lebesgue measurable function w; if w is a non-zero constant function, then $\mathcal{M}(X)$ are just the Lebesgue measurable subsets of X.

Assume that we start with $(X, \mathcal{M}(X), \mu)$. Next we are given subsets $\{X_\alpha \mid \alpha \in \mathcal{F}\}$ of X that satisfy the partition properties from Section 4. Such subsets usually are determined by the application; numerical solvers for integral and differential equations often recursively subdivide an original domain X into the X_α's, as illustrated in Example 8. Keeping with the previous notation, $\sigma(\{X_\alpha \mid \alpha \in \mathcal{F}\}) = \widetilde{\Sigma}$ and the μ-completion of $(X, \widetilde{\Sigma})$ is (X, Σ). We would like an easily verifiable condition on $\{X_\alpha \mid \alpha \in \mathcal{F}\}$ that would guarantee that $\Sigma = \mathcal{M}(X)$; for then, the corresponding wavelets form an unconditional basis for $\mathrm{L}_p(X, \mathcal{M}(X), \mu)$. Towards this, two useful measurement are the diameter of $S \subset \mathbf{R}^n$ given by

$$\operatorname{diam} S = \sup_{x,\, y \in S} d\,(x, y)$$

and the (μ-)essential diameter of $S \in \mathcal{M}(X)$ given by

$$\text{ess diam } S = \inf \{\operatorname{diam} Y \mid Y \subset S\,,\, Y \in \mathcal{M}(X)\,,\, \mu(S \setminus Y) = 0\}\,.$$

If X_α is in $\mathcal{B}(X)$ for each $\alpha \in \mathcal{F}$ and if for each $x \in X$

$$\inf_{\substack{\alpha \in \mathcal{F} \\ x \in X_\alpha}} \operatorname{diam} X_\alpha = 0\,, \tag{25}$$

then $\widetilde{\Sigma} = \mathcal{B}(X)$ and so $\Sigma = \mathcal{M}(X)$. It is possible to relax condition (25).

Proposition 17. *If X_α is in $\mathcal{M}(X)$ for each $\alpha \in \mathcal{F}$ and if for each $x \in X$*

$$\inf_{\substack{\alpha\in\mathcal{F}\\ x\in X_\alpha}} \textit{ess diam } X_\alpha = 0 \; , \tag{26}$$

then $\Sigma = \mathcal{M}(X)$.

As customary, for $y \in \mathbf{R}^n$ and $\epsilon > 0$, let

$$N_\epsilon(y) \;=\; \{x \in X \mid d(x,y) < \epsilon\} \; .$$

Proof. Since $\widetilde{\Sigma} \subset \mathcal{M}(X)$, it suffices to show that $\mathcal{B}(X) \subset \Sigma$. Towards this, fix any δ and ϵ with $0 < \delta < \epsilon$ and $y \in \mathbf{R}^n$. It suffices to find a set $N_\delta \in \Sigma$ so that $N_\delta(y) \subset N_\delta \subset N_\epsilon(y)$.

For each $x \in N_\epsilon(y)$, find *decreasing* sequences $\{X_{\alpha_n(x)}\}_n$ and $\{Y_{\alpha_n(x)}\}_n$ along with a sequence $\{y_{\alpha_n(x)}\}_n$ so that

1. $x \in X_{\alpha_n(x)}$
2. $y_{\alpha_n(x)} \in Y_{\alpha_n(x)} \subset X_{\alpha_n(x)}$
3. $\mu\left(X_{\alpha_n(x)} \setminus Y_{\alpha_n(x)}\right) = 0$
4. $\lim_n \operatorname{diam} Y_{\alpha_n(x)} = 0$
5. $\lim_n y_{\alpha_n(x)} = y_0(x) \quad$ for some $y_0(x) \in \mathbf{R}^n$.

If $d(y, y_0(x)) < \epsilon$, then pick $\alpha_n(x)$ so that

$$Y_{\alpha_n(x)} \subset N_\epsilon(y)$$

and if $d(y, y_0(x)) \geqslant \epsilon$, then pick $\alpha_n(x)$ so that

$$Y_{\alpha_n(x)} \subset [N_\delta(y)]^C \; .$$

Since

$$N_\epsilon(y) \subset \bigcup_{x\in N_\epsilon(y)} X_{\alpha_n(x)}$$

there is a countable subset $\{x_i \mid i \in J\}$ of $N_\epsilon(y)$ such that

$$N_\epsilon(y) \subset \bigcup_{i\in J} X_i$$

where $X_{\alpha_n(x_i)} = X_i$. Likewise, set $Y_{\alpha_n(x_i)} = Y_i$. Let $J_0 = \{i \in J \mid d(y, y_0(x_i)) < \epsilon\}$ and

$$N_\delta = \left[\bigcup_{i\in J} \quad (X_i \setminus Y_i) \;\cap\; N_\delta\,(y)\right] \quad \bigcup \quad \left[\bigcup_{i\in J_0} \quad Y_i\right] \; .$$

Then $N_\delta(y) \;\subset\; N_\delta \;\subset\; N_\epsilon(y)$ and $N_\delta \in \mathcal{M}(X)$. Thus $N_\epsilon(y) \in \Sigma$, as needed. $\square$

The use of the unbalanced Haar wavelets in applications is still somehow limited. The reason is that they are non-smooth and that they have only one vanishing moment; i.e. the integral of a wavelet vanishes, but the integral of a wavelet multiplied with a non-constant polynomial need not vanish. Consequently, the convergence of the expansion (12) is slow for a smooth function f. In [21, 22] the "lifting scheme" is described, which given one initial multiresolution analysis, can allows you to build a second, more performant one, in the sense that the wavelets have more vanishing moments or more smoothness. The Haar wavelets constructed in this paper are a perfect example for such an initial multiresolution analysis.

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