	$\widetilde{N}=2$		$\widetilde{N}=4$		$\widetilde{N}=6$
k	$\widetilde{h}_k$	k	$\widetilde{h}_k$	k	$\widetilde{h}_k$
-4	2^{-6}	-6	-2^{-9}	-8	$9 \cdot 2^{-14}$
-3	0	-5	0	-7	0
-2	-2^{-3}	-4	$9 \cdot 2^{-8}$	-6	$-35 \cdot 2^{-12}$
-1	$\mathbf{2^{-2}}$	-3	$\mathbf{-2^{-5}}$	-5	$\mathbf{9 \cdot 2^{-10}}$
0	$23 \cdot 2^{-5}$	-2	$-63 \cdot 2^{-9}$	-4	$189 \cdot 2^{-12}$
1	$\mathbf{2^{-2}}$	-1	$\mathbf{9 \cdot 2^{-5}}$	-3	$\mathbf{-59 \cdot 2^{-10}}$
2	-2^{-3}	0	$87 \cdot 2^{-7}$	-2	$-477 \cdot 2^{-12}$
3	0	1	$\mathbf{9 \cdot 2^{-5}}$	-1	$\mathbf{153 \cdot 2^{-9}}$
4	2^{-6}	2	$-63 \cdot 2^{-9}$	0	$5379 \cdot 2^{-13}$
		3	$\mathbf{-2^{-5}}$	1	$\mathbf{153 \cdot 2^{-9}}$
		4	$9 \cdot 2^{-8}$	2	$-477 \cdot 2^{-12}$
		5	0	3	$\mathbf{-59 \cdot 2^{-10}}$
		6	-2^{-9}	4	$189 \cdot 2^{-12}$
				5	$\mathbf{9 \cdot 2^{-10}}$
				6	$-35 \cdot 2^{-12}$
				7	0
				8	$9 \cdot 2^{-14}$

TABLE 2. Dual filter coefficients for $N = 4$. The lifting coefficients $s_k = 2\,\widetilde{h}_{1-2k}$ are boldfaced. One recognizes the Deslauriers-Dubuc coeffients of Table 1 in case $\widetilde{N} \leq N$.

	$\widetilde{N}=2$		$\widetilde{N}=4$		$\widetilde{N}=6$
k	$\widetilde{h}_k$	k	$\widetilde{h}_k$	k	$\widetilde{h}_k$
-6	$-3 \cdot 2^{-10}$	-8	$3 \cdot 2^{-13}$	-10	$-9 \cdot 2^{-17}$
-5	0	-7	0	-9	0
-4	11.2^{-9}	-6	$-13 \cdot 2^{-11}$	-8	$75 \cdot 2^{-16}$
-3	0	-5	0	-7	0
-2	$-125 \cdot 2^{-10}$	-4	$87 \cdot 2^{-11}$	-6	$-1525 \cdot 2^{-17}$
-1	$\mathbf{2^{-2}}$	-3	$\mathbf{-2^{-5}}$	-5	$\mathbf{3 \cdot 2^{-9}}$
0	181.2^{-8}	-2	$-243 \cdot 2^{-11}$	-4	$825 \cdot 2^{-14}$
1	$\mathbf{2^{-2}}$	-1	$\mathbf{9 \cdot 2^{-5}}$	-3	$\mathbf{-25 \cdot 2^{-9}}$
2	$-125 \cdot 2^{-10}$	0	$2721 \cdot 2^{-12}$	-2	$-7425 \cdot 2^{-16}$
3	0	1	$\mathbf{9 \cdot 2^{-5}}$	-1	$\mathbf{75 \cdot 2^{-8}}$
4	11.2^{-9}	2	$-243 \cdot 2^{-11}$	0	$21201 \cdot 2^{-15}$
5	0	3	$\mathbf{-2^{-5}}$	1	$\mathbf{75 \cdot 2^{-8}}$
6	$-3 \cdot 2^{-10}$	4	$87 \cdot 2^{-11}$	2	$-7425 \cdot 2^{-16}$
		5	0	3	$\mathbf{-25 \cdot 2^{-9}}$
		6	$-13 \cdot 2^{-11}$	4	$825 \cdot 2^{-14}$
		7	0	5	$\mathbf{3 \cdot 2^{-9}}$
		8	$3 \cdot 2^{-13}$	6	$-1525 \cdot 2^{-17}$
				7	0
				8	$75 \cdot 2^{-16}$
				9	0
				10	$-9 \cdot 2^{-17}$

TABLE 3. Dual filter coefficients for $N = 6$. The lifting coefficients $s_k = 2\,\widetilde{h}_{1-2k}$ are boldfaced. One recognizes the Deslauriers-Dubuc coeffients of Table 1 as indicated in Theorem 12.

[61] P. P. Vaidyanathan, T. Q. Nguyen, Z. Doğanata, and T. Saramäki. Improved technique for design of perfect reconstruction fir QMF banks with lossless polyphase matrices. *IEEE Trans. Acoust. Speech Signal Process.*, 37(7):1042–1055, 1989.

[62] M. Vetterli. Filter banks allowing perfect reconstruction. *Signal Process.*, 10:219–244, 1986.

[63] M. Vetterli. Running FIR and IIR filtering using multirate filter banks. *IEEE Trans. Signal Process.*, 36:730–738, 1988.

[64] M. Vetterli and D. Le Gall. Perfect reconstruction FIR filter banks: Some properties and factorizations. *IEEE Trans. Acoust. Speech Signal Process.*, 37:1057–1071, 1989.

[65] M. Vetterli and C. Herley. Wavelets and filter banks: theory and design. *IEEE Trans. Acoust. Speech Signal Process.*, 40(9):2207–2232, 1992.

[66] M. Vetterli and J. Kovačević. *Wavelets and Subband Coding*. Prentice Hall, Englewood Cliffs, NJ, 1995.

[67] G. G. Walter. *Wavelets and other orthogonal systems with applications*. CRC Press, Boca Raton, 1994.

[68] J. W. Woods and S. D. O'Neil. Subband coding of images. *IEEE Trans. Acoust. Speech Signal Process.*, 34(5):1278–1288, 1986.

Department of Mathematics, University of South Carolina, Columbia SC 29208, USA

Katholieke Universiteit Leuven, Belgium, Department of Computer Science

AT&T Bell Laboratories, 600 Mountain Avenue, Murray Hill, NJ 07974, USA *(current address)*

E-mail address: wim@research.att.com

[43] F. Mintzer. Filters for distortion-free two-band multirate filter banks. *IEEE Trans. Acoust. Speech Signal Process.*, 33:626–630, 1985.

[44] T. Q. Nguyen and P. P. Vaidyanathan. Two-channel perfect-reconstruction FIR QMF structures which yield linear-phase analysis and synthesis filters. *IEEE Trans. Acoust. Speech Signal Process.*, 37:676–690, 1989.

[45] O. Rioul and P. Duhamel. Fast algorithms for discrete and continuous wavelet transforms. *IEEE Trans. Inform. Theory*, 38(2):569–586, 1992.

[46] O. Rioul and M. Vetterli. Wavelets and signal processing. *IEEE Signal Proc. Mag.*, pages 14–38, October 1991.

[47] N. Saito and G. Beylkin. Multiresolution representations using the auto-correlation functions of compactly supported wavelets. *IEEE Trans. Signal Process.*, 41(12):3584–3590, 1993.

[48] P. Schröder and W. Sweldens. Spherical wavelets: Efficiently representing functions on the sphere. *Computer Graphics Proceedings, (SIGGRAPH 95)*, pages 161–172, 1995.

[49] M. J. Shensa. Wedding the à trous and Mallat algorithms. *IEEE Trans. Signal Process.*, 40(10):2464–2482, 1992.

[50] M. J. T. Smith and T. P. Barnwell. Exact reconstruction techniques for tree-structured subband coders. *IEEE Trans. Acoust. Speech Signal Process.*, 34(3):434–441, 1986.

[51] W. Sweldens. *Construction and Applications of Wavelets in Numerical Analysis*. PhD thesis, Department of Computer Science, Katholieke Universiteit Leuven, Belgium, 1994.
(`http://www.cs.att.com/math/people/wim/thesis/`).

[52] W. Sweldens. The lifting scheme: A construction of second generation wavelets. Technical Report 1995:6, Industrial Mathematics Initiative, Department of Mathematics, University of South Carolina, 1995.
(`ftp://ftp.math.sc.edu/pub/imi_95/imi95_6.ps`).

[53] W. Sweldens. The lifting scheme: A new philosophy in biorthogonal wavelet constructions. In A. F. Laine and M. Unser, editors, *Wavelet Applications in Signal and Image Processing III*, pages 68–79. Proc. SPIE 2569, 1995.

[54] W. Sweldens and R. Piessens. Quadrature formulae and asymptotic error expansions for wavelet approximations of smooth functions. *SIAM J. Numer. Anal.*, 31(4):1240–1264, 1994.

[55] W. Sweldens and P. Schröder. Building your own wavelets at home. Technical Report 1995:5, Industrial Mathematics Initiative, Department of Mathematics, University of South Carolina, 1995.
(`ftp://ftp.math.sc.edu/pub/imi_95/imi95_5.ps`).

[56] L. M. G. M. Tolhuizen, I. A. Shah, and A. A. C. M. Kalkar. On constructing regular filter banks from domain bounded polynomials. *IEEE Trans. Signal Process.*, 42(2):451–456, 1994.

[57] M. Unser. Efficient dyadic wavelet transformation of images using interpolating filters. In *Proc. Int. Conf. on Acoustics, Speech and Signal Processing, Minneapolis*, volume 5, pages 149–152, 1993.

[58] M. Unser, A. Aldroubi, and M. Eden. A family of polynomial spline wavelet transforms. *Signal Process.*, 30:141–162, 1993.

[59] P. P. Vaidyanathan. Theory and design of M-channel maximally decimated quadrature mirror filters with arbitrary M, having perfect reconstruction property. *IEEE Trans. Acoust. Speech Signal Process.*, 35(2):476–492, 1987.

[60] P. P. Vaidyanathan and P.-Q. Hoang. Lattice structures for optimal design and robust implementation of two-band perfect reconstruction QMF banks. *IEEE Trans. Acoust. Speech Signal Process.*, 36:626–630, Jan. 1988.

[22] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5(1):49–68, 1989.

[23] D. L. Donoho. Interpolating wavelet transforms. Preprint, Department of Statistics, Stanford University, 1992.

[24] P. Dutilleux. An implementation of the "algorithme à trous" to compute the wavelet transform. In [15], pages 298–304, 1989.

[25] M. Frazier and B. Jawerth. Decomposition of Besov spaces. *Indiana Univ. Math. J.*, 34(4):777–799, 1985.

[26] M. Frazier and B. Jawerth. A discrete transform and decompositions of distribution spaces. *J. Funct. Anal.*, 93:34–170, 1990.

[27] M. Girardi and W. Sweldens. A new class of unbalanced Haar wavelets that form an unconditional basis for L_p on general measure spaces. Technical Report 1995:2, Industrial Mathematics Initiative, Department of Mathematics, University of South Carolina, 1995. (`ftp://ftp.math.sc.edu/pub/imi_95/imi95_2.ps`).

[28] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The John Hopkins University Press, Baltimore, 2nd edition, 1983.

[29] A. Grossmann and J. Morlet. Decompostion of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.*, 15(4):723–736, 1984.

[30] C. Herley and M. Vetterli. Wavelets and recursive filter banks. *IEEE Trans. Signal Process.*, 41(8):2536–2556, 1993.

[31] M. Holschneider, R. Kronland-Martinet, J. Morlet, and Ph. Tchamitcian. A real-time algorithm for signal analysis with the help of the wavelet transform. In [15], pages 286–297, 1989.

[32] A. J. E. M. Janssen. The Zak transform and sampling theorems for wavelet subspaces. Preprint, Philips Research Laboratories, Eindhoven, The Netherlands.

[33] B. Jawerth and W. Sweldens. An overview of wavelet based multiresolution analyses. *SIAM Rev.*, 36(3):377–412, 1994.

[34] G. Kaiser. *A friendly guide to wavelets*. Birkhäuser, Boston, 1994.

[35] C. W. Kim, R. Ansari, and A. E. Cetin. A class of linear-phase regular biorthogonal wavelets. In *Proc. IEEE Int. Conference on Acoustics, Speech, Signal Processing*, pages 673–676, 1992.

[36] W. M. Lawton. Necessary and sufficient conditions for constructing orthonormal wavelets bases. *J. Math. Phys.*, 32(1):57–61, 1991.

[37] P. G. Lemarié. Some remarks on wavelet theory and interpolation. Technical Report 91-13, Mathématiques, Univ. Paris Sud, 1991.

[38] R. M. Lewis. Cardinal interpolating multiresolutions. *J. Approx. Theory*, 76:177–202, 1994.

[39] M. Lounsbery, T. D. DeRose, and J. Warren. Multiresolution surfaces of arbitrary topological type. Department of Computer Science and Engineering 93-10-05, University of Washington, October 1993. Updated version available as 93-10-05b, January, 1994.

[40] S. G. Mallat. Multifrequency channel decompositions of images and wavelet models. *IEEE Trans. Acoust. Speech Signal Process.*, 37(12):2091–2110, 1989.

[41] S. G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L_2(\mathbf{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, 1989.

[42] Y. Meyer. *Ondelettes et Opérateurs,* I: *Ondelettes,* II: *Opérateurs de Calderón-Zygmund,* III: (with R. Coifman), *Opérateurs multilinéaires*. Hermann, Paris, 1990. English translation of first volume, *Wavelets and Operators*, is published by Cambridge University Press, 1993.

References

[1] P. Abry and A. Aldroubi. Designing multiresolution analysis-type wavelets, and their fast algorithms. *J. Fourier Anal. Appl.*, 2(2):135–159, 1995.

[2] A. Aldroubi and M. Unser. Families of multiresolution and wavelet spaces with optimal properties. *Numer. Funct. Anal. Optim.*, 14:417–446, 1993.

[3] R. Ansari, C. Guillemot, and J. F. Kaiser. Wavelet construction using lagrange halfband filters. *IEEE Trans. Circuits Systems*, 38(9):1116–1118, 1991.

[4] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms I. *Comm. Pure Appl. Math.*, 44:141–183, 1991.

[5] J. M. Carnicer, W. Dahmen, and J. M. Peña. Local decompositions of refinable spaces. *Appl. Comput. Harmon. Anal.*, (to appear), 1996.

[6] D. Chen. Extended families of cardinal spline wavelets. *Appl. Comput. Harmon. Anal.*, 1:194–208, 1994.

[7] C. K. Chui. *An Introduction to Wavelets*. Academic Press, San Diego, CA, 1992.

[8] C. K. Chui and C. Li. Dyadic affine decompositions and functional wavelet transforms. *SIAM J. Math. Anal.*, to appear.

[9] C. K. Chui, L. Montefusco, and L. Puccio, editors. *Conference on Wavelets: Theory, Algorithms, and Applications*. Academic Press, San Diego, CA, 1994.

[10] C. K. Chui and X. Shi. Wavelets and multiscale interpolation. In T. Lyche and L. L. Schumaker, editors, *Mathematical Methods in Computer Aided Geometric Design II*, pages 111–133. Academic Press, 1992.

[11] C. K. Chui and J. Z. Wang. A cardinal spline approach to wavelets. *Proc. Amer. Math. Soc.*, 113:785–793, 1991.

[12] C. K. Chui and J. Z. Wang. A general framework of compactly supported splines and wavelets. *J. Approx. Theory*, 71(3):263–304, 1992.

[13] A. Cohen. Ondelettes, analyses multiresolutions et filtres miroirs en quadrature. *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, 7(5):439–459, 1990.

[14] A. Cohen, I. Daubechies, and J. Feauveau. Bi-orthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 45:485–560, 1992.

[15] J. M. Combes, A. Grossmann, and Ph. Tchamitchian, editors. *Wavelets: Time-Frequency Methods and Phase Space*. Inverse problems and theoretical imaging. Springer-Verlag, New York, 1989.

[16] W. Dahmen, S. Prössdorf, and R. Schneider. Multiscale methods for pseudo-differential equations on smooth manifolds. In [9], pages 385–424. 1994.

[17] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41:909–996, 1988.

[18] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conf. Series in Appl. Math., Vol. 61. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[19] I. Daubechies. Orthonormal bases of compactly supported wavelets II: Variations on a theme. *SIAM J. Math. Anal.*, 24(2):499–519, 1993.

[20] I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, 1986.

[21] G. Deslauriers and S. Dubuc. Interpolation dyadique. In *Fractals, dimensions non entières et applications*, pages 44–55. Masson, Paris, 1987.

As translation and dilation become algebraic operations in the frequency domain, the Fourier transform is the basic tool for the construction of classical wavelets. In the construction of second generation wavelets, the Fourier transform can no longer be used; thus, almost all traditional constructions fail. The real power of the lifting scheme lies in that fact that it can easily be generalized to construct compactly supported second generation wavelets. The basic idea is to choose a different set of coefficients s_k in (11) for each wavelet $\psi_{j,m}$. This way one can construct wavelets adapted to the cases (1)-(5) described above. None of the classical wavelet construction schemes allow for this. Again the lifting scheme requires an initial multiresolution analysis to start the construction. A Lazy wavelet transform can easily be found for all cases. In [27] a generalization of the Haar wavelets to arbitrary measure spaces, and thus cases (1)-(5), is given. This is another example of an initial second generation multiresolution analysis. The details of the second generation lifting scheme are described in [52]. One recent application is the construction of wavelets on a sphere [48]. In [55] several practical one-dimensional examples such as irregular samples, weights, and intervals, are worked out.

The basic motivation for the work in this paper was to gain a better understanding of the properties of the lifting scheme in the more familiar setting of classical wavelets in order to facilitate the generalization to the second generation case.

Acknowledgment. The work presented here benefited greatly from stimulating and inspiring discussions with Ingrid Daubechies, Anca Deliu, Maria Girardi, Björn Jawerth, Jelena Kovačević, Michael Unser, and Peter Schröder. Peter and Jelena also helped improving the exposition. Special thanks to Thomas Bueschgens for interesting e-mail discussions and for pointing out several typos in an earlier version.

Remark. After finishing this work, the author learned that related results were obtained independently by Wolfgang Dahmen and collaborators. Their results are stated in a setting similar to the second general wavelet case described above and lead to wavelets defined on manifolds and triangulations. We refer the reader to the original papers [5, 16] for details.

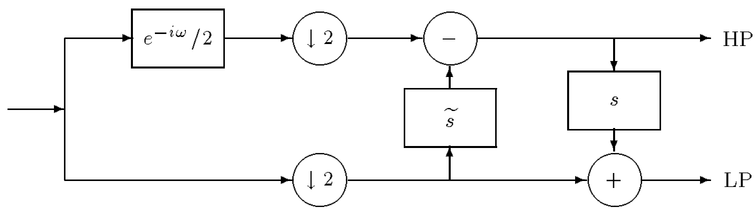

FIGURE 4. *The fast lifted wavelet transform in case of improved Donoho wavelets. First a Lazy wavelet transform, then a dual lifting, and finally a regular lifting. This is an example of a simple cakewalk.*

which we can use to start the lifting scheme. For more information on shift-interpolation, see [32, 54] and [51, Chapter 3].

4. The idea of shift-interpolation was also used by Chen [6] to construct extended families of semiorthogonal spline functions. In case $\tau \neq 0$ the dual functions in his family are no longer spline functions. As typical for the semiorthogonal case, the dual functions are not compactly supported.

9. CONCLUSION

As we mentioned before, in a regular setting the lifting scheme never leads to a wavelet that somehow could not be constructed before. The new features are the custom-design property and the speed-up of the fast wavelet transform. Another nice feature of lifting is that it allows fully in-place calculation of the wavelet transform. In other words, the memory locations of the original data can be overwritten by the wavelet coefficients without having to allocate new memory. In fact, the FFT has a similar feature which can be obtained by bit-reversing the addresses of the memory locations. In order to calculate the wavelet transform in place using lifting, a partial bit reversal is needed. More details can be found in [53].

The original motivation behind lifting, however, lied elsewhere, namely in the construction of second generation wavelets. The basic idea of second generation wavelets is to abandon translation and dilation to construct wavelets. The goal is to construct wavelets for spaces much more general than $L_2(\mathbf{R}, dx)$. Typical examples are

1. wavelets on an interval,
2. wavelets on domains in $\mathbf{R}^n$,
3. wavelets (bi)orthogonal with respect to a weighted inner product,
4. wavelets on curves, surfaces, and manifolds,
5. wavelets adapted to irregular sampling.

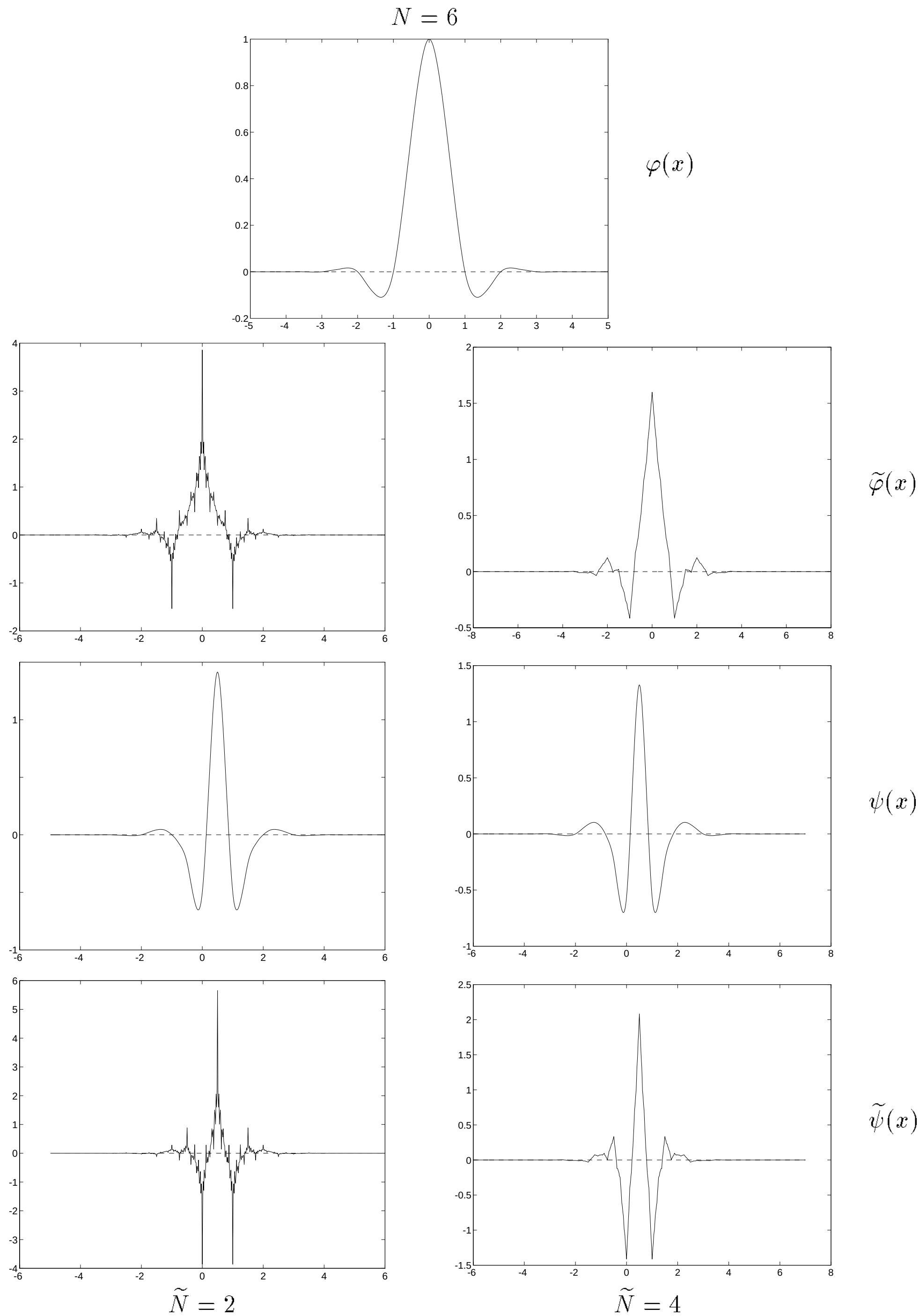

FIGURE 3. Scaling function and wavelets for $N = 6$.

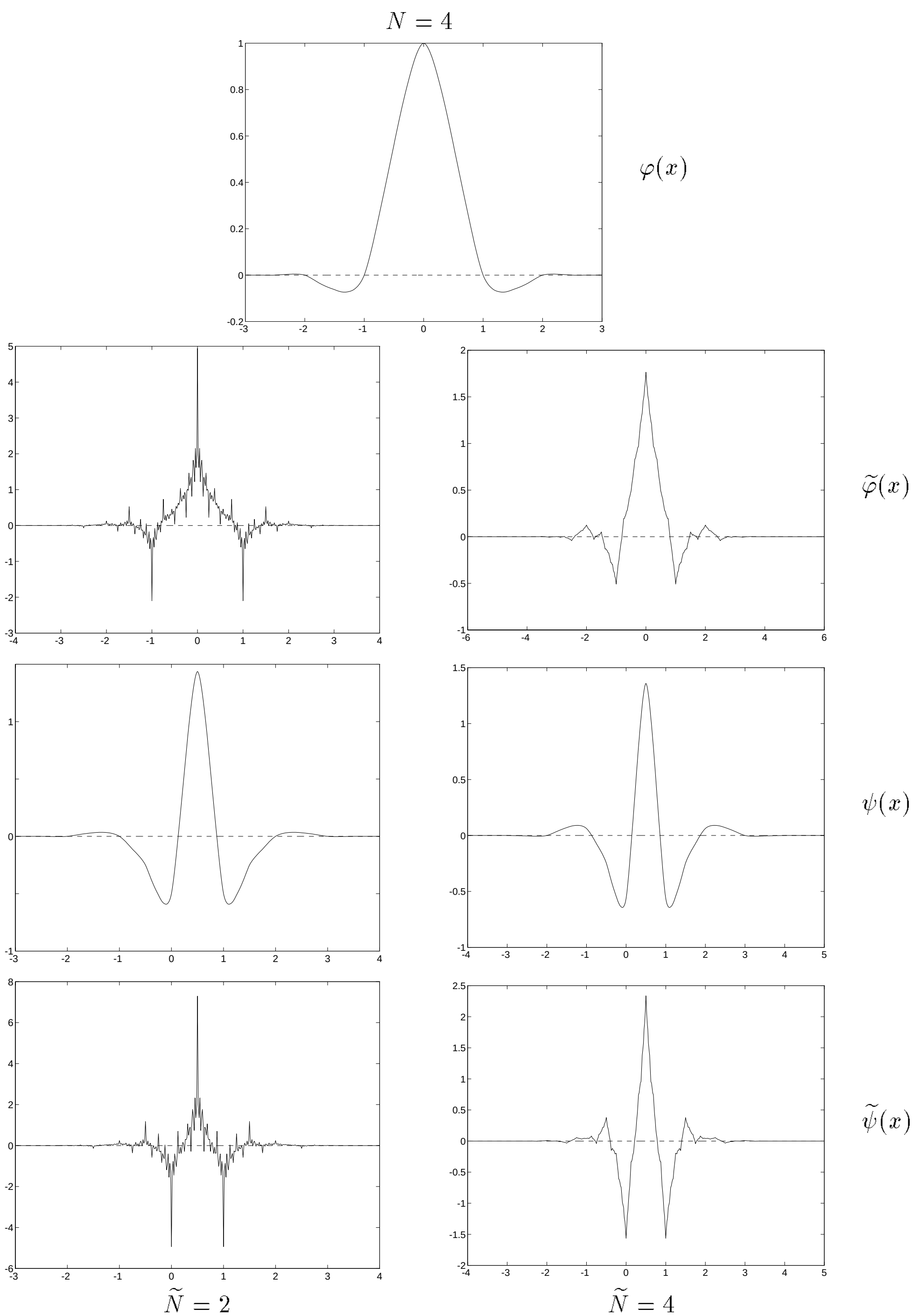

FIGURE 2. Scaling function and wavelets for $N = 4$.

[4, 54]. Thus, for all practical purposes one can take the coefficients to be the function values. This property was precisely the motivation behind the Donoho wavelets.

If $N = 2$, the wavelets coincide with the Cohen-Daubechies-Feauveau biorthogonal wavelets. Figures 2 and 3 give the graphs of the scaling function and wavelets for $N = 4, 6$ and $\widetilde{N} = 2, 4, 6$. One can see that the smoothness increases with increasing N and $\widetilde{N}$. For completeness, Tables 2 and 3 list the coefficients of $\widetilde{h}$ in the cases $N = 4, 6$ and $\widetilde{N} = 2, 4, 6$. Here you can see the effect of Theorem 12. The lifting coefficients can be found as $s_k = 2\widetilde{h}_{1-2k}$ and are shown in bold. For the cases where $\widetilde{N} \leq N$, one recognizes the Deslauriers-Dubuc coefficients as lifting coefficients.

Remarks:

1. Although the dual scaling functions with $\widetilde{N} = 2$ and $N > 2$ qualitatively resemble $\widetilde{\varphi}_{2,2}$, the dual scaling function of Cohen-Daubechies-Feauveau in case $\widetilde{N} = N = 2$, they are quantitatively quite different. If $N > 2$, the dual functions with $\widetilde{N} = 2$ are bounded everywhere. The dual function $\widetilde{\varphi}_{2,2}$, however, is discontinuous and actually becomes infinity at every dyadic point. This is somehow puzzling as the dyadics form a dense set. The function obviously cannot be graphed and can maybe serve as a text book example of how "ugly" a function in L_2 can be. The graph of $\widetilde{\varphi}_{2,2}$ shown in [18, p. 273] has to be thought of as one particular iteration of the cascade algorithm (which converges in L_2), see also [18, second edition, p. 287, note nr. 10]. The cascade algorithm for this function does not converge in L_∞. If one performs more iterations of the cascade algorithm, the spikes (corresponding to dyadic points) keep growing. The underlying reason is the degeneracy of 1 as an eigenvalue of the operator iterated in the cascade algorithm. In case $N > 2$, the cascade algorithm converges in L_∞.
2. The family of wavelets constructed here is closely connected to the filters constructed using Lagrange halfband filters in [35].
3. By generalizing the notion of interpolation to shift-interpolation, we can find more examples of initial sets of biorthogonal filters to start the lifting scheme. We define a scaling function φ to be shift-interpolating if a shift $\tau \in (0, 1)$ exists so that $\varphi(k + \tau) = \delta_{k,0}$. Again a characterization in terms of the h_k exists, since if φ is shift-interpolating with shift τ, then

$$\sum_l h_l \, \varphi(2\tau + 2k - l) = \delta_{k,0} \, .$$

This implies that we can find a dual filter $\widetilde{h}$ biorthogonal to h as

$$\widetilde{h}_k^0 = \varphi(2\tau + k) \, ,$$

The case $\widetilde{N} > N$: This case, which is less interesting, is more involved. We give only the basic outline of the construction. We use the same definition for d as in the proof above. Since N is even we also let $\widetilde{N}$ be even. The fact that g has a root of multiplicity $\widetilde{N}$ at the origin leads to

$$0 = \delta_{p,0} - \sum_{k=0}^{p} \binom{p}{k} h^{(k)}(0)\, d^{(p-k)}(0) \quad \text{for} \quad 0 \leq p < \widetilde{N} \,. \tag{22}$$

We let md_p and mh_p be the moments of the sequences $\{d_k\}$ and $\{h_k\}$ respectively,

$$md_p = \sum_k d_k\, k^p = i^p\, d^{(p)}(0) \,,$$

and similarly for mh_p. As d and h are even, all moments with odd index are zero. A recursion relation to calculate the md_p is now given by

$$md_{2p} = -\sum_{k=0}^{p-1} \binom{2p}{2k} md_{2k}\, mh_{2p-2k} \quad \text{for} \quad 1 \leq p < \widetilde{N}/2 \,.$$

The recursion starts with $d_0 = 1$, which follows from (22) in case $p = 0$. We let $\widetilde{N}$ of the coefficients d_k be nonzero, namely the ones where $k = 2l + 1$ and $-\widetilde{N}/2 \leq l < \widetilde{N}/2$. We can then find their values from

$$\sum_{l=0}^{\widetilde{N}/2-1} d_{2l+1}\, (2l+1)^{2p} = md_{2p}/2 \quad \text{for} \quad 0 \leq p < \widetilde{N}/2 \,.$$

This linear system has a $\widetilde{N}/2 \times \widetilde{N}/2$ Vandermonde matrix and can be solved in $\widetilde{N}^2/20$ operations, see [28, Algorithm 4.6.2, page 181]. It is now straightforward to find s, g and the dual functions.

With these two cases, one can construct an entire family of wavelets with interpolating scaling functions. One can check that if $\widetilde{N} > 0$, the conditions of Theorem 3 are satisfied and thus these wavelets generate dual Riesz bases for L_2. The associated forward wavelet transform now consists of the Lazy wavelet transform followed by dual lifting followed by lifting. A block diagram is depicted in Figure 4. The number of coefficients of s is about half the number of coefficients of $\widetilde{h}$. One can thus speed up the low-pass part of the fast wavelet transform compared to the standard algorithm by a factor of two.

From (19) and the fact that h has a root of order N at π it follows that $\widetilde{h} - 1$ has a root of order N at the origin as well. In other words, the dual scaling function also is a Coiflet of order N or

$$\int_r x^p\, \widetilde{\varphi}(x)\, dx = \delta_p \quad \text{for} \quad 0 \leq p < N \,. \tag{23}$$

In fact, it is not exactly the same as a Coiflet given that the number of moment conditions (23) (namely N) need not be the same as the order of polynomials reproduced (namely $\widetilde{N}$). One can think of it as a *biorthogonal Coiflet*. In case $\widetilde{N} \leq N$ this is precisely what one needs as the coefficients $\lambda_{n,k}$ can be numerically approximated with an error of $\mathcal{O}(2^{-nN})$ by using a simple one point quadrature formula where the first term on the right is precisely the Donoho wavelet. We can now choose the s_k coefficients to assure that ψ has $\widetilde{N}$ vanishing moments. In our construction we fix h to be the Deslauriers-Dubuc filter h^N, which can be written as

$$h^N(\omega) = 1/2 + e^{i\omega}\, \widetilde{s}^N(2\omega)\ . \tag{21}$$

We consider two cases $\widetilde{N} \leq N$ and $\widetilde{N} > N$ separately.

The case $\widetilde{N} \leq N$: This is the most common case. In image processing and numerical analysis the number of dual vanishing moments is much more important than the number of primal vanishing moments. Luckily it turns out that this case is by far the easiest. We want choose s so that ψ has $\widetilde{N}$ vanishing moments. The following theorem tells us precisely how to do this.

Theorem 12. *Consider the Deslauriers-Dubuc scaling function of order N and the Donoho wavelet that goes with it. If $\widetilde{N} \leq N$, lifting with*

$$s(\omega) = 2\, \widetilde{s}^{\widetilde{N}}(-\omega)$$

where $h^{\widetilde{N}}(\omega) = 1/2 + e^{i\omega}\, \widetilde{s}^{\widetilde{N}}(2\omega)$, results in the shortest wavelet (20) with $\widetilde{N}$ vanishing moments which is symmetric around 1/2.

For example in the case $\widetilde{N} = N$, the lifting coefficients are simply the dual lifting coefficients times two. In case $\widetilde{N} < N$, the lifting coefficients come from dual lifting coefficients of a lower order Deslauriers-Dubuc filter (namely $s^{\widetilde{N}}$).

Proof. Let $\widetilde{d}^N(\omega) = e^{-i\omega}\, \widetilde{s}_N(2\omega)$. Then $\widetilde{d}^N$ is the unique shortest filter that satisfies

- $\widetilde{d}^N(\omega) - 1/2$ has a root of order N at the origin,
- $\widetilde{d}^N$ has only odd taps: $\widetilde{d}^N(\omega) + \widetilde{d}^N(\omega + \pi) = 0$,
- $\widetilde{d}^N$ is symmetric: $\widetilde{d}^N(-\omega) = \widetilde{d}^N(\omega)$.

We need to find an s so that ψ has $\widetilde{N}$ vanishing moments. We first find a d of the form $d(\omega) = e^{i\omega}\, s(2\omega)$, and then later show that $d = 2\, \widetilde{d}^{\widetilde{N}}$. The theorem then follows from substituting d and $\widetilde{d}$ and using the fact that they are symmetric.

First, d obviously has only odd taps. Next, write

$$e^{i\omega} g(\omega) = 1 - h(\omega)\, d(\omega)\ .$$

In order to get a wavelet which is symmetric around 1/2, d has to be even. In order for the wavelet to have $\widetilde{N}$ vanishing moment g has a root of multiplicity $\widetilde{N}$ at the origin. If $d(0) = 1$ then $g(0) = 0$ since $h(0) = 1$. If $d^{(p)}(0) = 0$ for $1 \leq p < \widetilde{N}$, then $g^{(p)}(0) = 0$ because $h^{(p)}(0) = 0$. The shortest filter that satisfies these constraints is $d = 2\, \widetilde{d}^{\widetilde{N}}$. $\square$

There is a close connection between Donoho wavelets and the Lazy wavelet. Consider an interpolating filter h that satisfies (15). This implies that h can be written as

$$h(\omega) = 1/2 + e^{-i\omega}\, \widetilde{s}(2\omega) \ ,$$

where $\widetilde{s}$ is a trigonometric polynomial. Consequently,

$$\widetilde{g}(\omega) = e^{-i\omega}/2 - \overline{\widetilde{s}(2\omega)} \ .$$

Now these are precisely the equations we would get from applying the dual lifting scheme to the Lazy wavelet filters

$$2\, h^0(\omega) = \widetilde{h}^0(\omega) = 1 \quad \text{and} \quad g^{\,0}(\omega) = 2\, \widetilde{g}^0(\omega) = e^{-i\omega} \ .$$

(Note the slightly different normalization from the example given above.) Thus we have shown the following result.

Corollary 11. *The set of biorthogonal filters associated with an interpolating scaling function and the Dirac function as its dual can always be seen as the result of the dual lifting scheme applied to the Lazy wavelet filters.*

8.2. **Improving Donoho wavelets.** Donoho wavelets, however, in general can suffer from the following disadvantages.

1. They do not provide Riesz bases for L_2. The dual wavelets do not even belong to L_2. Another way to see this is that the wavelets do not have a vanishing integral and thus cannot form a Riesz basis for L_2, see [18, Chapter 3].
2. The fast wavelet transform introduces considerable aliasing. For example the low pass filter $\widetilde{h}(\omega)$ used in the fast wavelet transform is simply a constant.

These wavelets are useful in case the function one wishes to expand is smooth, which is precisely the setting in [23].

In this section, we try to overcome these disadvantages. We start from the set of biorthogonal filters associated with an interpolating scaling function and the Dirac as its dual, cf. (18). The lifting scheme then results in

$$\widetilde{h}(\omega) = 1 + e^{-i\omega}\, \overline{h(\omega+\pi)}\, \overline{s(2\omega)} \ , \quad \text{and} \quad g(\omega) = e^{-i\omega} - h(\omega)\, s(2\omega) \ . \tag{19}$$

Note that

$$\psi(x) = 2\, \varphi(2x-1) - \sum_k s_k\, \varphi(x-k) \ , \tag{20}$$

$N=2$		$N=4$		$N=6$		$N=8$	
k	h^2_k	k	h^4_k	k	h^6_k	k	h^8_k
-1	2^{-2}	-3	-2^{-5}	-5	$3\cdot 2^{-9}$	-7	$-5\cdot 2^{-12}$
0	2^{-1}	-2	0	-4	0	-6	0
1	2^{-2}	-1	$9\cdot 2^{-5}$	-3	$-25\cdot 2^{-9}$	-5	$49\cdot 2^{-12}$
		0	2^{-1}	-2	0	-4	0
		1	$9\cdot 2^{-5}$	-1	$75\cdot 2^{-8}$	-3	$-245\cdot 2^{-12}$
		2	0	0	2^{-1}	-2	0
		3	-2^{-5}	1	$75\cdot 2^{-8}$	-1	$1225\cdot 2^{-12}$
				2	0	0	2^{-1}
				3	$-25\cdot 2^{-9}$	1	$1225\cdot 2^{-12}$
				4	0	2	0
				5	$3\cdot 2^{-9}$	3	$-245\cdot 2^{-12}$
						4	0
						5	$49\cdot 2^{-12}$
						6	0
						7	$-5\cdot 2^{-12}$

TABLE 1. Deslauriers-Dubuc filters h^N for $N = 2, 4, 6, 8$.

These filters are given in Table 1. For $N = 2$ the associated scaling function is the Hat function or linear B-spline. The scaling functions are always continuous and the regularity grows asymptotically for large N as $.2075N$ [18, p. 226].

8. WAVELETS WITH INTERPOLATING SCALING FUNCTIONS

In this section we show how dual lifting connects Donoho wavelets to the Lazy wavelet and how lifting can be used to improve Donoho wavelets.

8.1. **Donoho wavelets.** Given a filter h associated with an interpolating scaling function, we can find a trivial set of biorthogonal filters by letting

$$\widetilde{h}^0(\omega) = 1, \qquad g^0(\omega) = e^{-i\omega}, \quad \text{and} \quad \widetilde{g}(\omega) = e^{-i\omega}\,\overline{h(\omega+\pi)}\,. \tag{18}$$

It is easy to see that (4) is satisfied. An associated set of biorthogonal functions formally exists. The dual scaling function is the Dirac impulse at the origin and the wavelet function becomes

$$\psi(x) = 2\,\varphi(2x-1)\,.$$

The dual wavelet is a linear combination of Dirac impulses since

$$\widehat{\widetilde{\psi}}(\omega) = e^{-i\omega}\,\overline{h(\omega+\pi)}\,.$$

Donoho introduced these wavelets [23], where he shows they form unconditional bases for certain smoothness spaces. We therefore call them *Donoho wavelets.* Chui and Li use similar wavelets [8] to define functional wavelet transforms, a setting in which duals need not be in L_2.

The construction (17), when started from the Daubechies orthogonal or Cohen-Daubechies-Feauveau biorthogonal wavelets, yields a family of smooth, compactly supported, and interpolating functions which were studied originally by Deslauriers and Dubuc [21, 22] and which are also known as Lagrange halfband filters [3]. This connection was made in [3, 49, 47]. Ansari et al [3] show how the Daubechies filters can be derived from Lagrange halfband filters, Shensa [49] points out a connection between the fast wavelet transform and the à trous algorithm, and Saito and Beylkin [47] use the autocorrelation function to build a multiresolution analysis in which they relate the zero-crossings to the locations of edges at different scales in the signal. They then reconstruct a signal from its zero-crossings and the slopes at the zero-crossings. Vetterli and Herley [65] also use this connection in the construction of biorthogonal wavelets.

Using (17), statements concerning (bi)orthogonal scaling functions can be rephrased as statements concerning interpolating scaling functions. This allows us to formulate a necessary and sufficient condition on h for φ to be interpolating using the ideas of Theorem 3, see also [13] and [18, Chapter 6].

Theorem 10. *The following two conditions are equivalent:*

1. *φ defined by (7) is in L_2 and is interpolating,*
2. *h is a trigonometric polynomial with $h(0) = 1$ and $h(\omega) + h(\omega + \pi) = 1$, and the constants are the only invariant trigonometric polynomials under R, where R acts on 2π-periodic functions as*

$$(R\,a)(\omega) = h(\omega/2)\,a(\omega/2) + h(\omega/2 + \pi)\,a(\omega/2 + \pi)\ .$$

This condition precisely excludes counterexamples such as (16). Although it appears technical it can be checked easily by looking at the eigenvalues of a matrix H with entries $H_{k,l} = h_{2l-k}$. The matrix approach in the orthogonal case was introduced by Lawton [36].

It immediately follows from (15) that $h(\omega) - 1$ has a root of order N at the origin. In other words, interpolating scaling functions always have the Coiflet [19] property in the sense that

$$\int_{-\infty}^{+\infty} x^p\,\varphi(x)\,dx = \delta_p \quad \text{for} \quad 0 \le p < N\ .$$

We next consider the family of interpolating filter and scaling functions derived by Deslauriers and Dubuc [21]. The Deslauriers-Dubuc filters are indexed by an even parameter N and we denote them as h^N. They are the shortest filters that combine the following properties:

- h^N is interpolating: $h^N(\omega) + h^N(\omega + \pi) = 1$,
- h^N is symmetric: $h^N(-\omega) = h^N(\omega)$,
- the scaling functions reproduce polynomials up to degree N: or h^N has a root of order N at π.

7. Interpolating scaling functions

The use of interpolating scaling functions has been proposed by several authors, see [2, 8, 10, 23, 37, 38, 47, 57]. In the next section, we show how one can use the lifting scheme to construct compactly supported interpolating scaling functions. We therefore first recall some results concerning interpolating scaling functions in this section.

Definition 8. A scaling function φ is interpolating if $\varphi(k) = \delta_k$ for all $k \in \mathbf{Z}$.

The advantage of using an interpolating scaling functions is that the coefficients of an expansion

$$f(x) = \sum_k \lambda_k \, \varphi(x-k) \; ,$$

satisfy $f(k) = \lambda_k$. The following well-known proposition allows us to characterize interpolating scaling functions.

Proposition 9. *If φ is interpolating, then $h_{2k} = \delta_{k,0}/2$ for all $k \in \mathbf{Z}$.*

This condition can also be written as

$$h(\omega) + h(\omega + \pi) = 1 \; . \tag{15}$$

Note that the converse is not always true. The following function is a typical counterexample:

$$\varphi(x) = \begin{cases} 3+x & \text{for } \; x \in [-3, 0) \\ 3-x & \text{for } \; x \in [0, 3) \\ 0 & \text{elsewhere.} \end{cases} \tag{16}$$

It is a stretched hat function with $h_{-3} = h_3 = 1/4$, $h_0 = 1/2$ and the other h_k zero, and is not interpolating.

We refer to filters that satisfy condition (15) as *interpolating filters*. They are also known as *à-trous filters* which are used in the *à trous algorithm*, a method to quickly compute samples of a continuous wavelet transform, see e.g. [24, 31, 45, 49].

There is a close connection between a pair of biorthogonal scaling functions and an interpolating scaling function. More precisely, if φ and $\widetilde{\varphi}$ are biorthogonal then Φ defined as

$$\Phi(x) = \int_{\infty}^{\infty} \varphi(y) \, \widetilde{\varphi}(y+x) \, dy \tag{17}$$

is interpolating and vice versa. The interpolation property immediately follows from the biorthogonality condition. In case φ is an orthogonal scaling function Φ is simply its autocorrelation function.

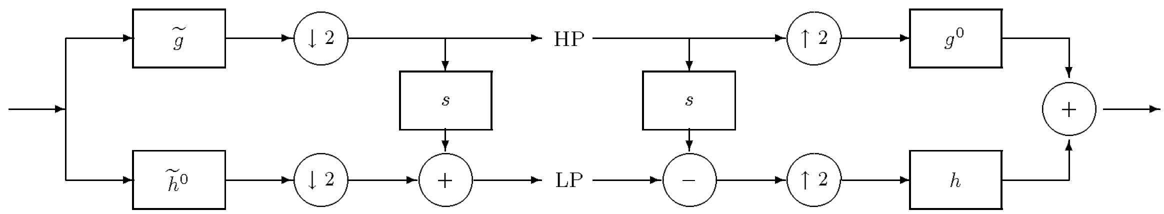

FIGURE 1. *The lifted fast wavelet transform: The basic idea is to first perform a classical subband filter with simple filters and later "lifting" the lower subband with the help of the higher subband. In the case of dual lifting the higher subband would be lifted with the help of the lower one.*

faster implementation (again compared to the standard case) of the wavelet transform if we let $\widetilde{h}^0$ to be the shortest filter that is biorthogonal to h, and think of the given $\widetilde{h}$ as a lifting from $\widetilde{h}^0$.

Consider the example of Section 4.1. The low-pass step of the fast wavelet transform in the standard case is

$$\lambda_{j,l} = \sqrt{2}\,(-1/16\,\lambda_{j+1,2l-2} + 1/16\,\lambda_{j+1,2l-1} + 1/2\,\lambda_{j+1,2l} + 1/2\,\lambda_{j+1,2l+1} + 1/16\,\lambda_{j+1,2l+2} - 1/16\,\lambda_{j+1,2l+3})\ .$$

Given that $s_1 = -s_{-1} = 1/8$, this now becomes

$$\lambda_{j,l} = \sqrt{2}\,(1/2\,\lambda_{j+1,2l} + 1/2\,\lambda_{j+1,2l+1}) + 1/8\,\gamma_{j,l-1} - 1/8\,\gamma_{j,l+1}\ ,$$

which has fewer operations. Obviously, during implementation, one also absorbs the factor $\sqrt{2}$ in the coefficients and takes advantage of the symmetry. If $\widetilde{g}$ is longer, a greater reduction in operations can be obtained.

As we mentioned, the standard algorithm is not necessarily the best way to implement the wavelet transform. Exploiting the special structure of the filters after lifting is only one idea in a whole tool bag of methods to improve the speed of a fast wavelet transform. Another idea consists of factoring the filters, as suggested in [18, Section 6.4]. This can be combined with lifting. It involves factoring $h(\omega)$, $\widetilde{h}(\omega)$, and $s(\omega)$ over the reals where possible. Rioul and Duhamel discuss several other schemes to improve the standard fast wavelet transform [45]. In the case of long filters, they suggest an FFT based scheme known as the Vetterli-algorithm [65]. In the case of short filters, they suggest a "fast running FIR" algorithm [63]. How these ideas combine with the idea of using lifting and which combination will be optimal for a certain wavelet remains a topic of future research. The main point of this section is to show that keeping the lifting structure of the filters can be beneficial.

h, $\widetilde{h}^0$, g^0, $\widetilde{g}$ and s. This can reduce the number of operations as compared to the standard algorithm. Using (12), we can write the first equation of (14) as

$$\lambda_{j,l} = \sqrt{2} \sum_k \widetilde{h}^0_{k-2l}\, \lambda_{j+1,k} + \sum_k s_{l-k}\, \gamma_{j,k} \ .$$

This assumes that we calculate the $\gamma_{j,l}$ coefficients before the $\lambda_{j,l}$. Each step of the forward transform thus splits into two stages: (I) a classical subband splitting using the simple filters $\widetilde{h}^0$ and $\widetilde{g}$, (II) an update of the low subband $\{\lambda_{j,l}\}$ using the s filter on the high subband $\{\gamma_{j,k}\}$. For the inverse transform we obtain by using (10) that

$$\lambda_{j+1,k} = \sqrt{2} \sum_l h_{k-2l} \left(\lambda_{j,l} - \sum_m s_{l-m}\, \gamma_{j,m} \right) + \sqrt{2} \sum_l g^0_{k-2l}\, \gamma_{j,l} \ .$$

Here stage I is simply undoing the stage II of the forward transform, while stage II is a classical subband merging using the simple filters h and g^0. This leads to the following algorithm for the *fast lifted wavelet transform*. A block scheme is depicted in Figure 1. Implementation of dual lifting or cakewalk (alternating lifting and dual lifting) is now straightforward.

Forward transform:

- **Stage I:** (Calculate the unlifted coefficients)

$$\lambda_{j,l} := \sqrt{2} \sum_k \widetilde{h}^0_{k-2l}\, \lambda_{j+1,k} \quad \text{and} \quad \gamma_{j,l} := \sqrt{2} \sum_k \widetilde{g}_{k-2l}\, \lambda_{j+1,k} \ .$$

- **Stage II:** (Calculate the lifted coefficients)

$$\lambda_{j,l} := \lambda_{j,l} + \sum_k s_{l-k}\, \gamma_{j,k} \ .$$

Inverse transform:

- **Stage I:**

$$\lambda_{j,l} := \lambda_{j,l} - \sum_k s_{l-k}\, \gamma_{j,k} \ .$$

- **Stage II:**

$$\lambda_{j+1,k} := \sqrt{2} \sum_l h_{k-2l}\, \lambda_{j,l} + \sqrt{2} \sum_l g^0_{k-2l}\, \gamma_{j,l} \ .$$

Assume that we are in the situation where the filters h and $\widetilde{h}$ are given. Is there a way to speed up the wavelet transform associated with these filters? Remember that the Vetterli-Herley lemma tells us that *any* filter $\widetilde{h}^0$ that is biorthogonal to h is related to $\widetilde{h}$ through the lifting scheme. We thus get a

5. We can use the lifting scheme to do other things than just increase the number of vanishing moments. One idea is to choose s to get better frequency resolution and less aliasing. Another idea is to shape the wavelet for use in feature recognition. Choose the s_k coefficients in (11) so that ψ resembles the particular feature we want to recognize. The magnitude of the wavelet coefficients now is proportional to how much the signal at the particular scale and place resembles the feature. This has important applications in automated target recognition and medical imaging.
6. The idea of custom designing a wavelet was also suggested by Aldroubi and Unser [2], Abry and Aldroubi [1] and Chui and Wang [12]. They introduce several schemes to control the shape, support, regularity, and interpolating properties of the wavelet. The difference with our approach is that we work in the fully biorthogonal case, while they work in the semiorthogonal case. The latter has the advantage that the wavelet and scaling function are orthogonal to each other. The disadvantage is that the dual filters are not guaranteed to be finite.
7. One can also use the lifting scheme to construct semiorthogonal wavelets. One then needs to choose the s_k in (11) so that $\varphi(\cdot - l)$ and ψ are orthogonal. This in general leads to infinite filters.

6. The fast wavelet transform

For a function $f \in \mathrm{L}_2$, define

$$\lambda_{j,l} = \langle f, \widetilde{\varphi}_{j,l} \rangle \quad \text{and} \quad \gamma_{j,l} = \langle f, \widetilde{\psi}_{j,l} \rangle \ ,$$

where $\widetilde{\psi}_{j,l}$ and $\widetilde{\varphi}_{j,l}$ are defined similarly to $\psi_{j,l}$. Consider the following problem: given the $\lambda_{n,l}$ for a fixed n, calculate the wavelet coefficients $\gamma_{j,l}$ for $j < n$. This is classically done through recursive applications of the formulae

$$\lambda_{j,l} = \sqrt{2} \sum_k \widetilde{h}_{k-2l}\, \lambda_{j+1,k} \quad \text{and} \quad \gamma_{j,l} = \sqrt{2} \sum_k \widetilde{g}_{k-2l}\, \lambda_{j+1,k} \ . \tag{14}$$

The inverse transform recursively uses the formula

$$\lambda_{j+1,k} = \sqrt{2} \sum_l h_{k-2l}\, \lambda_{j,l} + \sqrt{2} \sum_l g_{k-2l}\, \gamma_{j,l} \ .$$

The resulting algorithm is linear and is known as the fast wavelet transform. It is exactly the same as subband filtering. We use this algorithm, which need not be optimal, as a comparison basis and refer to it as the standard algorithm. It is the same as what is called the basic algorithm in [45, Section III.B] in the sense that it avoids calculating filtered coefficients that will be immediately subsampled.

If the filters are constructed with lifting, we can take advantage of this in the fast wavelet transform. Essentially, we never need to explicitly construct the filters $\widetilde{h}$ and g, but instead we can always work with does the job. Consequently,

$$\widetilde{h}(\omega) = -1/32\, e^{3i\omega} + 9/32\, e^{i\omega} + 1/2 + 9/32\, e^{-i\omega} - 1/32\, e^{-3i\omega} \,.$$

This implies that $\widetilde{\varphi}$ is the Deslauriers-Dubuc interpolating scaling function of order 4 [22]. The scaling function φ is still the Dirac impulse.

5. Discussion

1. Evidently, we can also go from an initial set of filters $\{h^0, \widetilde{h}, g, \widetilde{g}^{\,0}\}$ to $\{h, \widetilde{h}, g, \widetilde{g}\}$ thereby changing the filters h and $\widetilde{g}$ while keeping $\widetilde{h}$ and g unchanged. We denote the trigonometric polynomial involved by $\widetilde{s}(\omega)$. This operation will lead to the *dual lifting scheme*. Relationships like (11), (12), and (13) can be obtained by simply toggling the tildes. In case of dual lifting the dual scaling function remains unchanged, while the primal scaling function and dual wavelet change. The primal wavelet also changes, but again in a much less fundamental way as the coefficients of the refinement relation (g_k) remain the same. Dual lifting can be used to improve the properties of the dual wavelet or equivalently the primal scaling function.
2. One fascinating aspect is that lifting can be iterated. For example, after increasing the number of vanishing moments of the wavelet, one can use the dual lifting scheme to increase the number of vanishing moments of the dual wavelet. By alternating lifting and dual lifting, one can bootstrap one's way up to a multiresolution analysis with desired properties. We call this a "cakewalk" construction. For more details we refer to [52].
3. The lifting scheme never yields a set of filters that somehow could not be found before. Essentially every set of filters constructed using the lifting scheme could also have been derived using the machinery of Daubechies, Cohen-Daubechies-Feauveau, or Vetterli-Herley. For example, lifting the Lazy wavelet results in trying to find an even function t so that $t(\omega) + t(\omega + \pi) = 0$ and $1/2 - t(\omega)$ has as a root of a certain multiplicity at the origin. This is precisely the same problem from which Daubechies starts in her construction of orthogonal wavelets (in her notation $m_0(\omega) = 1/2 - t(\omega + \pi)$). This leads to a solution involving a combinatorial expression.
4. New insights coming from the lifting scheme are threefold. First, we can use equation (11) to get immediate access to the wavelet function and choose the filter s so that it satisfies certain properties. This allows painless custom design of the wavelet. Second, we show in the next section that the lifting scheme can speed up the wavelet transform. Finally, the lifting scheme allows for construction of second generation wavelets which are not translates and dilates of one fixed function. None of the traditional wavelet construction schemes allow this generalization.

After lifting $\{h, \widetilde{h}^0, g^0, \widetilde{g}\}$ to $\{h, \widetilde{h}, g, \widetilde{g}\}$ we obtain

$$g(\omega) = g^0(\omega) - h(\omega)\, s(2\omega)\ .$$

In order to have one vanishing moment we need $g(0) = 0$ and thus $s(0)$ has to be 0. Having two vanishing moments is equivalent to $g'(0) = 0$, which results in

$$g^{0\prime}(0) = h(0)\, s'(0)\, 2 + h'(0)\, s(0)\ .$$

This implies that $s'(0) = -i/4$. In order to keep the symmetry, we choose $s(\omega) = -i/4 \sin\omega$. Consequently,

$$\widetilde{h}(\omega) = -1/16\, e^{2i\omega} + 1/16\, e^{i\omega} + 1/2 + 1/2\, e^{-i\omega} + 1/16\, e^{-2i\omega} - 1/16\, e^{-3i\omega}\ .$$

It turns out that this is precisely one of the biorthogonal filters of Cohen-Daubechies-Feauveau [14, 18].

4.2. **Lifting the Lazy wavelet.** The Lazy wavelet leads to a fascinating example of an initial set of biorthogonal filters. It is essentially a set of biorthogonal filters that *do not do anything.* More precisely,

$$2\, \widetilde{h}^0(\omega) = h^0(\omega) = 1 \quad \text{and} \quad \widetilde{g}^{\,0}(\omega) = 2\, g^0(\omega) = e^{-i\omega}\ .$$

One step in a Lazy wavelet transform is nothing else but subsampling into the even and odd indexed samples. In fact, in this case of regularly spaced samples, the Lazy wavelet is nothing new. It is exactly the same as the *polyphase representation* often used in the design of filter banks, see for example [59, 61]. The reason why we gave it its own name is that it is instrumental in case of irregular samples and second generation wavelets [52].

An associated set of biorthogonal functions $\{\varphi^0, \widetilde{\varphi}^0, \psi^0, \widetilde{\psi}^0\}$ does not exist in L_2. Formally one can think of φ^0 as a Dirac impulse at the origin and of $\widetilde{\varphi}^0$ as a function that is one in the origin and zero elsewhere. Then φ^0 and $\widetilde{\varphi}^0$, again purely formally, would be biorthogonal. Needless to say, they can never form Riesz bases.

After lifting $\{h^0, \widetilde{h}^0, g^0, \widetilde{g}^{\,0}\}$ to $\{h^0, \widetilde{h}, g, \widetilde{g}^{\,0}\}$ we obtain

$$g(\omega) = e^{-i\omega}/2 - s(2\omega)\ .$$

Given that $e^{i\omega}\, g^0(\omega)$ is even and assuming we want to keep symmetry, we need to choose s so that $e^{i\omega}\, s(2\omega)$ is even. We call the latter function $t(\omega)$, so

$$e^{i\omega}\, g(\omega) = 1/2 - t(\omega)\ .$$

We choose t so that the left-hand side has a root of order 4 at the origin. Simple calculations show that the even function

$$t(\omega) = 9/16\, \cos\omega - 1/16\, \cos 3\omega$$

Although formally similar, the expressions in (11) and (12) are quite different. The difference lies in the fact that in (11) the scaling functions on the right-hand side did not change after lifting, while in (12) the functions on the right-hand side did change after lifting. Equation (12) does not really give much insight into how the dual scaling function changes and therefore is not much help in the choice of the s_k. On the other hand, (11) tells us precisely what happens to the wavelet after lifting and will be the key to finding the s_k coefficients. The dual wavelet (13) also changes, but in a much less fundamental way than the wavelet and dual scaling function. More precisely, the dual wavelet changes because the dual scaling functions from which it is built change, while the coefficients of the linear combination ($\widetilde{g}_k$) remain exactly the same.

The power behind the lifting scheme is that through s we have full control over all wavelets and dual functions that can be built from a particular scaling function. This means we can start from a simple or trivial set of biorthogonal functions and use (11) to choose s so that the wavelet after lifting has some desirable properties. This allows custom design of the wavelet and it is the motivation behind the name "lifting scheme." Since the scaling functions on the right-hand side of (11) do not change after lifting, conditions on ψ immediately translate into conditions on s. For example, we can choose s to increase the number of vanishing moments of the wavelet, or choose s such that ψ resembles a particular shape.

The advantage of using (11) as opposed to (2) for the construction of ψ is that in the former we have total freedom in the choice of s. Once s fixed, the lifting scheme assures that all filters are finite and biorthogonal. If we used (2) to construct ψ, we would have to check the biorthogonality separately. The lifting scheme thus allows us, to isolate into s, the degrees of freedom that are left after fixing the biorthogonality conditions.

Evidently, the lifting scheme is useful only if we have an initial set of biorthogonal filters. The following section gives two examples of initial sets.

4. Examples

4.1. **Lifting the Haar wavelet.** We start from the Haar wavelet and try to use the lifting scheme to increase the number of vanishing moments of the wavelet from one to two. We have

$$\widetilde{h}^0(\omega) = h(\omega) = 1/2 + 1/2\, e^{-i\omega} ,$$

and

$$\widetilde{g}(\omega) = g^0(\omega) = -1/2 + 1/2\, e^{-i\omega} .$$

Note that because of (5), we have that $g^0(\pi) = -1$, while most authors prefer to have $g^0(\pi) = 1$.

wavelet as,

$$\begin{aligned}
\widehat{\psi}(\omega) &= g(\omega/2)\,\widehat{\varphi}(\omega/2) \\
&= g^0(\omega/2)\,\widehat{\varphi}(\omega/2) - s(\omega)\,h(\omega/2)\,\widehat{\varphi}(\omega/2) \\
&= g^0(\omega/2)\,\widehat{\varphi}(\omega/2) - s(\omega)\,\widehat{\varphi}(\omega)\ .
\end{aligned}$$

This means that

$$\psi(x) = 2\sum_k g_k^0\,\varphi(2x-k) - \sum_k s_k\,\varphi(x-k)\ .$$

Similarly we obtain

$$\begin{aligned}
\widehat{\widetilde{\varphi}}(\omega) &= \widetilde{h}(\omega/2)\,\widehat{\widetilde{\varphi}}(\omega/2) \\
&= \widetilde{h}^0(\omega/2)\,\widehat{\widetilde{\varphi}}(\omega/2) + \overline{s(\omega)}\,\widetilde{g}(\omega/2)\,\widehat{\widetilde{\varphi}}(\omega/2) \\
&= \widetilde{h}^0(\omega/2)\,\widehat{\widetilde{\varphi}}(\omega/2) + \overline{s(\omega)}\,\widehat{\widetilde{\psi}}(\omega)\ .
\end{aligned}$$

Thus,

$$\widetilde{\varphi}(x) = 2\sum_k \widetilde{h}_k^0\,\widetilde{\varphi}(2x-k) + \sum_k s_{-k}\,\widetilde{\psi}(x-k)\ .$$

We can summarize these observations in the following theorem.

Theorem 7 (Lifting scheme)**.** *Take an initial set of biorthogonal scaling functions and wavelets* $\{\varphi,\,\widetilde{\varphi}^0,\,\psi^0,\,\widetilde{\psi}^0\}$*. Then a new set* $\{\varphi,\,\widetilde{\varphi},\,\psi,\,\widetilde{\psi}\}$*, which is formally biorthogonal can be found as*

$$\psi(x) = \psi^0(x) - \sum_k s_k\,\varphi(x-k) \tag{11}$$

$$\widetilde{\varphi}(x) = 2\sum_k \widetilde{h}_k^0\,\widetilde{\varphi}(2x-k) + \sum_k s_{-k}\,\widetilde{\psi}(x-k) \tag{12}$$

$$\widetilde{\psi}(x) = 2\sum_k \widetilde{g}_k\,\widetilde{\varphi}(2x-k) \tag{13}$$

where the coefficients s_k *can be freely chosen.*

Note that the resulting functions are only formally biorthogonal. Indeed, it is not guaranteed that the new dual wavelets belong to L_2 or that the new wavelets form a Riesz basis. For each choice of s, the conditions of Theorem 3 have to be verified, in order to assure that an associated set of biorthogonal functions exists. The only conditions to verify are the non-degeneracy of 1 as an eigenvalue of $\widetilde{P}$ and the sign of the invariant trigonometric polynomial. If s and the initial filters are finite, all associated basis and dual functions are compactly supported. An interesting question is whether the conditions of Theorem 3 can be translated into a simple condition on s.

where $s(\omega)$ is a trigonometric polynomial. Conversely, if one of the dual filters is biorthogonal to h, and they are related through (8)*, the other one is biorthogonal to h as well.*

Proof. The converse statement follows immediately from combining (6) and (8) and using the fact that $s(\omega)$ is 2π-periodic.

The proof of the first statement follows the reasoning in [18, p. 133]. Theorem 3 implies that two positive constants A and B exist so that

$$A < |h(\omega)|^2 + |h(\omega+\pi)|^2 < B \, .$$

Consequently, $h(\omega)$ and $h(\omega+\pi)$ cannot vanish together. Now let $d = \widetilde{h} - \widetilde{h}^0$, then

$$d(\omega)\,\overline{h(\omega)} + d(\omega+\pi)\,\overline{h(\omega+\pi)} = 0 \, .$$

It follows that $d(\omega) = e^{-i\omega}\overline{h(\omega+\pi)}\,s(2\omega)$, where $s(\omega)$ is a trigonometric polynomial. □

After finishing this work, the author learned that this lemma is essentially identical to an earlier proposition of Vetterli and Herley [65, Proposition 4.7]. We therefore will refer to Lemma 5 as the *Vetterli-Herley lemma*. It turns out that the same lemma was also used for the construction of filter banks in [56] and in [35].

The Vetterli-Herley lemma gives us a complete characterization of all filters biorthogonal to a given filter. The following corollary now immediately follows from using (5).

Corollary 6. *Take an initial set of finite biorthogonal filters $\{h, \widetilde{h}^0, g^0, \widetilde{g}\}$. Then a new set of finite biorthogonal filters $\{h, \widetilde{h}, g, \widetilde{g}\}$ can be found as*

$$\widetilde{h}(\omega) = \widetilde{h}^0(\omega) + \widetilde{g}(\omega)\,\overline{s(2\omega)} \tag{9}$$

$$g(\omega) = g^0(\omega) - h(\omega)\,s(2\omega) \, , \tag{10}$$

where $s(\omega)$ is a trigonometric polynomial.

We will refer to this procedure as *lifting* for reasons that will become clear later. We next investigate how this procedure affects the basis functions. This will allow us to precisely formulate lifting.

First we note that the associated scaling function does not change after lifting, since it depends only on the filter h. The associated dual scaling function, wavelet, and dual wavelet do change. We write the

Given a set of finite biorthogonal filters, it is not guaranteed that an associated set of biorthogonal functions exists. The following theorem from [14], see also [18, Theorem 8.3.1], gives a necessary and sufficient condition. We first define the operator P acting on 2π-periodic functions as

$$(P\,a)(\omega) = |h(\omega/2)|^2\, a(\omega/2) + |h(\omega/2+\pi)|^2\, a(\omega/2+\pi)\ ,$$

and similarly for $\widetilde{P}$.

Theorem 3 (Cohen-Daubechies-Feauveau). *Given two finite filters h and $\widetilde{h}$, the following statements are equivalent:*

1. *φ and $\widetilde{\varphi}$ defined by the product expansion (7) are in L_2 and $\langle\, \varphi, \widetilde{\varphi}(\cdot - l)\,\rangle = \delta_l$,*
2. *the finite filters h and $\widetilde{h}$ are biorthogonal in the sense of (6), $h(0) = \widetilde{h}(0) = 1$, 1 is a non-degenerate eigenvalue of P and of $\widetilde{P}$, and the invariant trigonometric polynomials are strictly positive.*

If we now choose g as in (5), and ψ as in (2), then the $\psi_{j,l}$ form a Riesz basis for L_2.

There are many different ways in which the high pass filters can be derived from the low pass filters. A complete characterization of all choices is given in a theorem of Chui [7, Theorem 5.19, p. 148]. We state the theorem using the notations introduced above.

Theorem 4 (Chui). *Let h and $\widetilde{h}$ be biorthogonal filters in the sense of (6). Then the filters g and $\widetilde{g}$ satisfy (4) if and only they are chosen from the class:*

$$\widetilde{g}(\omega) = e^{-i\omega}\, \overline{h(\omega+\pi)}\, k(2\omega) \quad \text{and} \quad g(\omega) = e^{-i\omega}\, \overline{\widetilde{h}(\omega+\pi)\, k^{-1}(2\omega)}\ ,$$

where k belongs to the Wiener class and $k(\omega) \neq 0$.

A filter belongs to the Wiener class if its coefficients sequence is in ℓ^1. If $k(\omega)$ does not vanish, $k^{-1}(\omega)$ is in the Wiener class as well. While this theorem gives a complete characterization, it also shows that the high pass filters g and $\widetilde{g}$ can both be finite only in case k is a monomial. In the remainder of this paper, we restrict ourselves to the latter case.

3. The lifting scheme

We start with the following observation.

Lemma 5. *Fix a compactly supported scaling function φ, and let h be the finite filter associated with it. Consider two finite dual filters $\widetilde{h}$ and $\widetilde{h}^0$, both of them biorthogonal to h in the sense of (6) and satisfying the conditions of Theorem 3. Then they are related to each other by*

$$\widetilde{h}(\omega) = \widetilde{h}^0(\omega) + e^{-i\omega}\, \overline{h(\omega+\pi)}\, \overline{s(2\omega)}\ , \tag{8}$$

and similarly for $\widetilde{m}(\omega)$. In the signal processing literature these matrices are called *modulation matrices*. In the case of finite filters, $\det m(\omega)$ is a monomial. We choose the determinant as

$$\det m(\omega) = -e^{-i\omega} ,$$

so that

$$\widetilde{g}(\omega) = e^{-i\omega}\, \overline{h(\omega+\pi)} \quad \text{and} \quad g(\omega) = e^{-i\omega}\, \overline{\widetilde{h}(\omega+\pi)} \ . \tag{5}$$

Then (4) is equivalent to

$$\overline{h(\omega)}\, \widetilde{h}(\omega) + \overline{h(\omega+\pi)}\, \widetilde{h}(\omega+\pi) = 1 \ . \tag{6}$$

This condition in the orthogonal case ($h = \widetilde{h}$) is called the *Smith-Barnwell condition* as they used it to design the first orthogonal filter banks [50].

Let N be the number of vanishing moments of the dual wavelet,

$$\int_{-\infty}^{+\infty} x^p\, \widetilde{\psi}(x)\, dx = 0 \quad \text{for} \quad 0 \le p < N \ .$$

It is also the multiplicity of the origin as a root of $\widetilde{g}(\omega)$. Similarly, let $\widetilde{N}$ be the number of vanishing moments of the wavelet. In a multiresolution analysis, $\widetilde{N}$ and N are at least 1. The scaling function and its integer translates can reproduce any polynomial of degree strictly less than N. We therefore say that the order of the multiresolution analysis is N.

Define the Fourier transform of a function f as

$$\widehat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx \ .$$

By iterating the refinement relation, we can write the Fourier transform of a scaling function as

$$\widehat{\varphi}(\omega) = \prod_{j=1}^{\infty} h(2^{-j}\omega) \ , \tag{7}$$

where the product converges absolutely and uniformly on compact sets. A similar statement holds for the dual scaling function $\widetilde{\varphi}$.

Definition 1. The set of functions $\{\varphi,\, \widetilde{\varphi},\, \psi,\, \widetilde{\psi}\}$ is a *set of biorthogonal functions* if conditions (3) are satisfied.

Definition 2. The set of filters $\{h,\, \widetilde{h},\, g,\, \widetilde{g}\,\}$ is a *set of finite biorthogonal filters* if condition (4) is satisfied and $\det m(\omega) = -e^{-i\omega}$.

2. Multiresolution analysis

A *multiresolution analysis* of $L_2(\mathbf{R})$ is built using two basis functions: a scaling function φ and a wavelet ψ. The scaling function $\varphi \in L_2$ satisfies a refinement relation in the sense that,

$$\varphi(x) = 2 \sum_k h_k \, \varphi(2x - k) \, . \tag{1}$$

The integer translates of the scaling function $\{\varphi(x-k) \mid k \in \mathbf{Z}\}$, form a Riesz basis for the closure of their span. They also partition the unity as

$$\sum_k \varphi(x-k) = 1 \, .$$

The wavelet function $\psi \in L_2$ is given by the refinement relation

$$\psi(x) = 2 \sum_k g_k \, \varphi(2x - k) \, . \tag{2}$$

The functions $\psi_{j,l}(x) = \sqrt{2^j}\, \varphi(2^j x - l)$, with $j,\, l \in \mathbf{Z}$ form a Riesz basis of L_2.

The dual scaling function $\widetilde{\varphi}$ and wavelet $\widetilde{\psi}$ also generate a multiresolution analysis. They satisfy refinement relations like (1) and (2) with coefficients $\widetilde{h}_k$ and $\widetilde{g}_k$ respectively. They are biorthogonal to φ and ψ in the sense that

$$\langle\, \widetilde{\varphi}, \psi(\cdot - l)\,\rangle \;=\; \langle\, \widetilde{\psi}, \varphi(\cdot - l)\,\rangle \;=0 \quad \text{and} \quad \langle\, \widetilde{\varphi}, \varphi(\cdot - l)\,\rangle \;=\; \langle\, \widetilde{\psi}, \psi(\cdot - l)\,\rangle \;= \delta_l \, . \tag{3}$$

Define the 2π-periodic functions

$$h(\omega) = \sum_k h_k \, e^{-ik\omega} \quad \text{and} \quad g(\omega) = \sum_k g_k \, e^{-ik\omega} \, ,$$

and similarly for the dual functions. We refer to h as the filter associated with φ, where we can think of h either as the sequence of coefficients $\{h_k \mid k\}$, or as the 2π-periodic function $h(\omega)$. We consider only the case where the scaling function, wavelet, and their duals are compactly supported. Consequently, only a finite number of the coefficients in the refinement relations are non-zero while $h(\omega)$ and $g(\omega)$ are trigonometric polynomials. In this setting, we refer to h and g as finite filters (also known as FIR — **f**inite **i**mpulse **r**esponse — filters).

A necessary condition for the biorthogonality (3) is

$$\forall \omega \in \mathbf{R} \;:\; \widetilde{m}(\omega)\, \overline{m^t(\omega)} = 1 \, , \tag{4}$$

where

$$m(\omega) = \begin{bmatrix} h(\omega) & h(\omega+\pi) \\ g(\omega) & g(\omega+\pi) \end{bmatrix} \, ,$$

In this paper, we always work in the fully biorthogonal setting and consider only the case where the scaling function, wavelet, and their duals are compactly supported. We present the *lifting scheme*, a novel way of looking at the construction of biorthogonal wavelets that allows custom design of the wavelet. In wavelet constructions one typically needs to simultaneously satisfy two groups of constraints: the biorthogonality relations (A) and various other constraints such as regularity, vanishing moments, frequency localization, and shape (B). The lifting scheme relies on a simple relationship between all multiresolution analyses that share the same scaling function. It thus isolates the degrees of freedom left after fixing the biorthogonality relations (A). Then one has full control over the remaining degrees of freedom to satisfy (B) and custom design the wavelet. Once the wavelet is defined, a compactly supported biorthogonal dual wavelet and scaling function immediately follow from the lifting scheme. Essentially, one can choose a particular scaling function, form a trivial multiresolution analysis with it, and use the lifting scheme to bootstrap one's way up to a multiresolution analysis with specific properties. This is the motivation behind the name "lifting scheme." In a translation/dilation invariant setting, the lifting scheme will not come up with wavelets that somehow could not be found using the techniques developed by Cohen-Daubechies-Feauveau [14] or Vetterli-Herley [65]. The new insights from lifting are: (i) custom design of wavelets, (ii) a new idea to speed up the wavelet transform, (iii) a generalization to non translation/dilation invariant settings (second generation wavelets).

The paper is organized as follows. In Section 2 we give a concise introduction to multiresolution analysis. We essentially list only the properties we later use. We refer the reader who is not familiar with this subject matter to more detailed treatments such as [7, 18, 33, 34, 42, 46, 66, 67]. In Section 3 we state the basic result, which is illustrated with examples in Section 4. The following section contains a discussion, while in Section 6 we show how the lifting scheme can speed up the implementation of the fast wavelet transform. In Section 7 we review some properties of interpolating scaling functions. We use the lifting scheme to build a family of wavelets from interpolating scaling functions in Section 8. Finally, we conclude by showing how this result fits in a broader line of research.

The results in this paper are inspired by the work [23] and [39]. Donoho [23] suggests the idea of wavelets built from interpolating scaling functions, while the wavelets of Lounsbery et al. [39] can be seen as a particular instance of the lifting scheme in case one wants to construct semiorthogonal wavelets.

THE LIFTING SCHEME: A CUSTOM-DESIGN CONSTRUCTION OF BIORTHOGONAL WAVELETS

WIM SWELDENS

November 1994 (revised November 1995)

ABSTRACT. We present the lifting scheme, a new idea for constructing compactly supported wavelets with compactly supported duals. The lifting scheme uses a simple relationship between all multiresolution analyses with the same scaling function. It isolates the degrees of freedom remaining after fixing the biorthogonality relations. Then one has full control over these degrees of freedom to custom design the wavelet for a particular application. The lifting scheme can also speed up the fast wavelet transform. We illustrate the use of the lifting scheme in the construction of wavelets with interpolating scaling functions.

1. INTRODUCTION

Over the last few years many constructions of wavelets have been introduced both in the mathematical analysis and in the signal processing literature. In fact the fruitful interaction between these communities is largely responsible for the success of wavelets. In mathematical analysis, wavelets were originally constructed to analyze and represent geophysical signals using translates and dilates of one fixed function. A mathematical framework was developed by the so-called "French school" [15, 20, 29, 42], see also [25, 26]. In signal processing, wavelets originated in the context of subband coding, or more precisely quadrature mirror filters [43, 44, 50, 59, 60, 61, 62, 64, 68]. The connection between the two approaches was made by the introduction of multiresolution analysis and the fast wavelet transform by Mallat and Meyer in [40, 41, 42]. A major breakthrough was the construction of orthogonal, compactly supported wavelets by Daubechies [17]. Since then, several generalizations to the biorthogonal or semiorthogonal (pre-wavelet) case were presented. (In the latter case wavelets on different levels are orthogonal, while wavelets on the same level are not.) Biorthogonality allows the construction of symmetric wavelets and thus linear phase filters. Examples are: the construction of semiorthogonal spline wavelets [2, 7, 11, 12, 58], fully biorthogonal compactly supported wavelets [14, 65], and recursive filter banks [30].

1991 *Mathematics Subject Classification.* 42C15.

Key words and phrases. wavelet, biorthogonal, multiresolution analysis, fast wavelet transform.

The author was partially supported by NSF EPSCoR Grant EHR 9108772 and DARPA Grant AFOSR F49620-93-1-0083. He is also Senior Research Assistant of the National Fund of Scientific Research Belgium (NFWO).