NORMAL MULTIRESOLUTION APPROXIMATION OF CURVES

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February 2002

ABSTRACT. A multiresolution analysis of a curve is normal if each wavelet detail vector with respect to a certain subdivision scheme lies in the local normal direction. In this paper we study properties such as regularity, convergence, and stability of a normal multiresolution analysis. In particular we show that these properties critically depend on the underlying subdivision scheme and that in general the convergence of normal multiresolution approximations equals the convergence of the underlying subdivision scheme.

1. INTRODUCTION

Subdivision is a powerful mechanism for iteratively *creating* smooth curves and surfaces. Combined with wavelets, subdivision can be used to *approximate* arbitrary functions, curves and surfaces. The mathematical properties of wavelets are well understood in the so-called "functional setting", i.e., for the approximation of *functions* of one of more variables. However, for the case of 1-D curves in the plane, or 2-D surfaces in 3-space, much less is known. Typically one takes a parameterization of the original curve or surface and ends up using wavelet analysis in each of the two or three components. This means the wavelet coefficients now become 2- or 3-vectors. It is important to choose an appropriate coordinate frame to describe these wavelet vectors. It is known that using an absolute coordinate frame for the wavelet or detail vectors leads to undesirable effects when editing curves; using a local coordinate frame defined by the normal works much better, as shown in [9, 8, 10, 16, 20].

In [11] the notion of normal approximation for curves or surfaces is introduced. A multiresolution approximation of a curve or surface is *normal* if all the wavelet vectors perfectly align with a locally defined normal direction which only depends on the coarser levels. Note that by the normal direction we mean a normal onto an approximation of the curve or surface. Given that this normal direction only depends on coarser levels, only a *single* scalar coefficient needs to be stored instead of the standard 2- or 3-vector. This is clearly extremely useful for compression applications, see [14, 13]. In addition, [11] gives an algorithm to build normal mesh approximations of large complex scanned geometry.

Because they depend on the computation of a normal, these approximations lead to non-linear representations and very little is known about their mathematical properties. In this paper we investigate in detail mathematical properties, such as convergence, regularity, and stability of normal multiresolution approximation for curves. In particular we show that these properties critically depend on the underlying subdivision scheme and that in general the convergence of the normal approximation of smooth curves equals the convergence of the subdivision scheme. Our central idea is to study normal approximation as a perturbation of a linear subdivision scheme.

The organization of this paper is as follows. After the preliminary Section 2, which sets notation and recalls some basic definitions and properties of subdivision schemes, we outline our main theorems in Section 3, explaining how they tie in with each other, leading up to our main results. In these initial formulations, we typically state the theorems in a more readable but slightly less general or technical form than later in the paper. The next three sections contain the technical part of the paper, with statements of the theorems in full generality, together with their proofs. Section 4 studies in some detail perturbations of sequences produced by linear subdivision schemes, and estimates how much applying a smooth function can perturb subdivided sequences; these results are used in

1991 *Mathematics Subject Classification.* 42C40, 65D10, 65D17, 65T60, 65U07.

Key words and phrases. subdivision, wavelet, normal mesh, normal multiresolution, lifting.

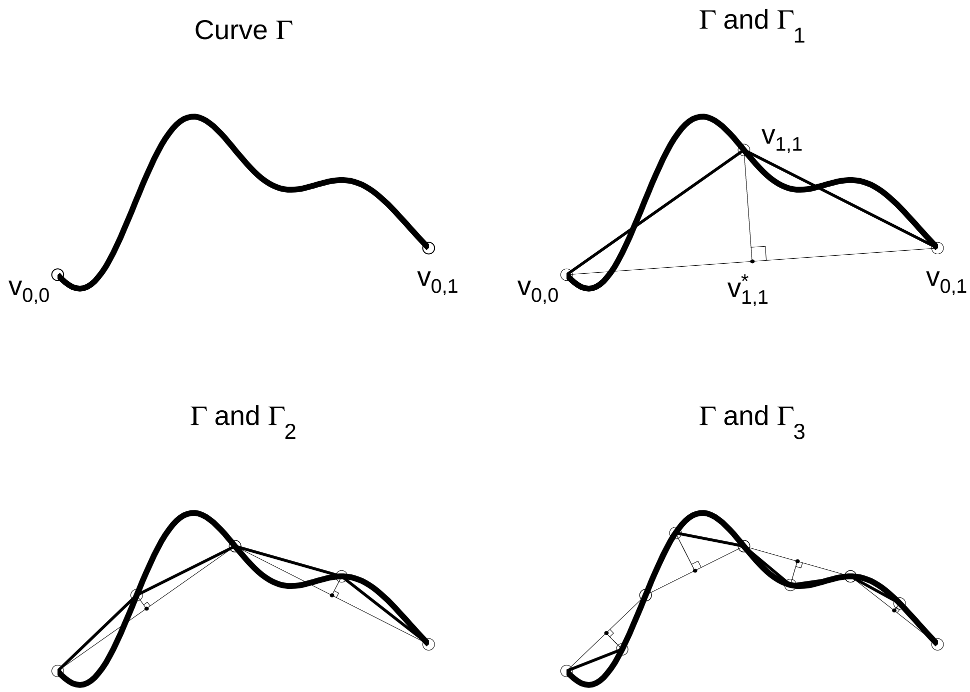

FIGURE 1. Example of the normal mesh algorithm using the mean value of adjacent points as predictor.

the remainder of the paper but may have wider applications. In Section 5, we discuss convergence of the normal multiresolution approximation. Section 6 relates the speed of convergence and the rate of decay of the wavelet coefficients with the degree of smoothness of the curve and the approximation order of the underlying subdivision scheme. Section 7 gives examples; Section 8 outlines several remaining open questions.

2. NOTATION AND PRELIMARIES

2.1. **Normal multiresolution analysis.** Figure 1 illustrates the main idea from [11] in the case of a normal approximation based on midpoint subdivision. The original curve Γ is described by successively finer approximations, which are organized in different *multiresolution layers* indexed by j. We assume that Γ is a continuous, non intersecting curve in the plane, whose endpoints we shall take to be the 0th level multiresolution points $v_{0,0}$ and $v_{0,1}$. To construct the vertices at level $(j+1)$, we first set $v_{j+1,2k} = v_{j,k}$; this is what makes the construction interpolating. We also compute new points $v_{j+1,2k+1}$; each $v_{j+1,2k+1}$ lies in between the two old points $v_{j,k}$ and $v_{j,k+1}$. This is done by first using subdivision to compute a predicted or base point $v^*_{j+1,2k+1}$. In the case of Figure 1 we use simply midpoint subdivision given by $v^*_{j+1,2k+1} = (v_{j,k} + v_{j,k+1})/2$. We next draw a line from $v^*_{j+1,2k+1}$ in the direction orthogonal to the line segment $(v_{j,k}, v_{j,k+1})$. This line is guaranteed to cross the curve segment between $v_{j,k}$ and $v_{j,k+1}$ at least once and we call one of those points $v_{j+1,2k+1}$. As this procedure continues, the polyline Γ_j, i.e. the piecewise linear curve connecting the points $v_{j,k}$ comes closer and closer to Γ. We can now think of this as a wavelet transformation similar to the notion of lifting [18]. Think of $v^*_{j+1,2k+1}$ as a prediction of the real point $v_{j+1,2k+1}$ computed based only on coarser information. Then the difference $v^*_{j+1,2k+1} - v_{j,2k+1}$ is a wavelet vector. Given that this vector points in a direction normal to a segment that again only depends on coarser data, we only need to store the length and one sign bit for this normal component to characterize it completely.

We shall be interested in using more general subdivision schemes, which will lead to higher quality approximation for smooth curves, as we shall see below. As illustrated in Figure 2, the same basic plan is followed: we still

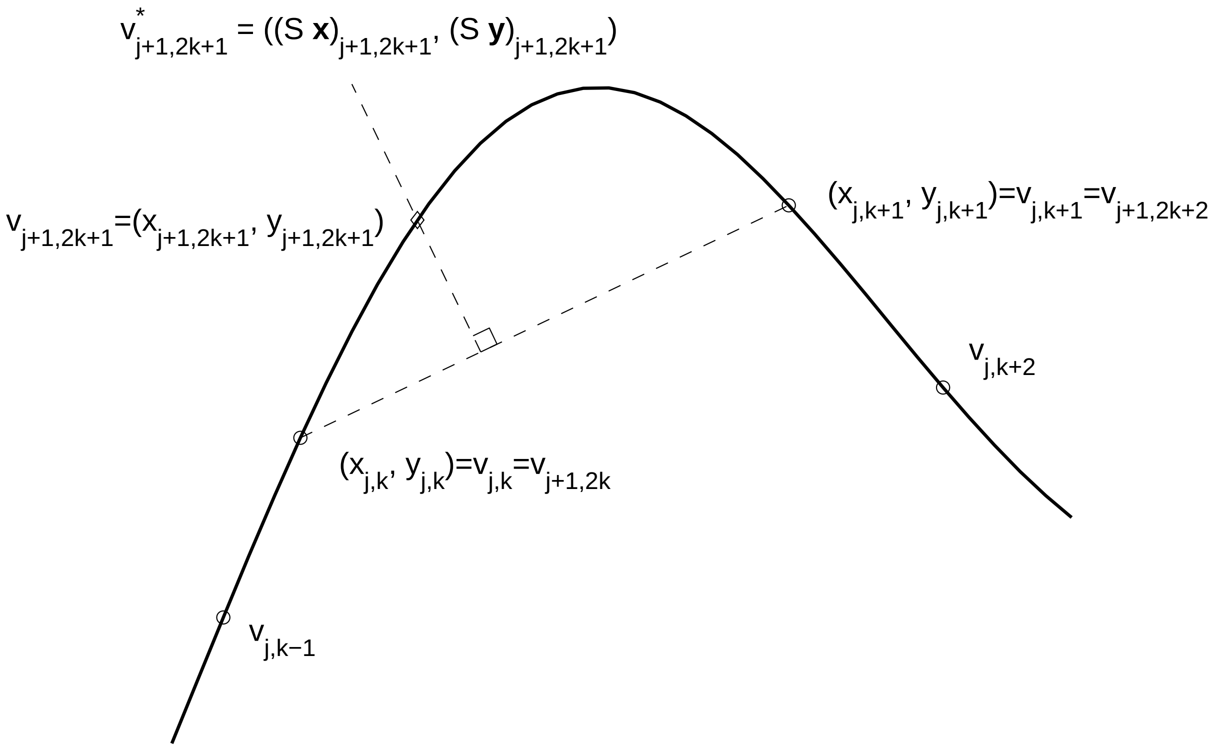

FIGURE 2. Notation for the normal scheme.

set $v_{j+1,2k} = v_{j,k}$, we define $v^*_{j+1,2k+1}$ via a subdivision scheme S (see below), and we define $v_{j+1,2k+1}$ to be an intersection point between $v_{j,k}$ and $v_{j,k+1}$ of Γ with the normal through $v^*_{j+1,2k+1}$ to the segment $(v_{j,k}, v_{j,k+1})$. We again define the *polyline* Γ_j to be the piecewise linear curve that connects each $v_{j,k+1}$ with its "predecessor" $v_{j,k}$. The construction immediately begs the following question: how good an approximation to Γ is is the *polyline* Γ_j? In other words, how does the distance between Γ and the j-th level polyline Γ_j decay as j tends to ∞? It turns out that the answer is given by the regularity of the subdivision scheme used in the prediction step of the normal approximation algorithm; the precise study of this dependence is our main topic. Note that in a normal approximation every $v_{j+1,2k+1}$ depends in a nonlinear way on the $(v_{j,k})$; these nonlinear aspects complicate the analysis. Nevertheless, due to the smoothness of Γ, this nonlinear map can be viewed as a perturbation of the underlying linear subdivision scheme used for predicting the $v^*_{j+1,k}$ from the $(v_{j,k})$; this is the key observation on which our analysis is based.

2.2. **Sequences.** We let X denote the space of infinite sequences. Sequences will be written in bold face, and elements of sequences in normal font, $\boldsymbol{x} := (x_k)_k$ or simply (x_k). We define the difference operator Δ as

$$(\Delta \boldsymbol{x})_k = x_{k+1} - x_k. \tag{2.1}$$

Often a sequence itself is indexed by the subdivision level j; then we use the convention that $\boldsymbol{x}_j := (x_{j,k})$. We think of a sequence at level j as associated with the parameters $t_{j,k} = k2^{-j}$. Therefore we also define the divided difference operator $D_j = 2^j\Delta$. The divided differences of a sequence $\boldsymbol{x}_j$ then are

$$\boldsymbol{x}_j^{[p]} = D_j^p \boldsymbol{x} \qquad p > 0.$$

We use the usual sup-norm on X, $|\boldsymbol{x}|_\infty = \sup_k |x_k|$. Scalar functions can be applied to sequences component wise, so that $(F(\boldsymbol{x}))_k = F(x_k)$. We use the special sequence $\boldsymbol{k} = (k)$, i.e. the sequence of which the k-th entry is k itself. The sequence with all entries equal to 0 (resp. 1) is $\mathbf{0}$ (resp. $\mathbf{1}$).

2.3. **Subdivision.** A local, stationary subdivision scheme is characterized by a bounded linear operator S from X to itself, defined by a finite sequence $\boldsymbol{s}$ as follows:

$$(S\boldsymbol{x})_k = \sum_\ell s_{k-2\ell} x_\ell.$$

The width B of S is defined by $B = 2\max\{|k|; s_k \neq 0\}$. The above sum thus is finite: for each k there are only terms with $l \in I_k = [\lceil (k-B)/2) \rceil, \lfloor (k+B)/2 \rfloor]$. Given S, we can apply it iteratively starting from a sequence $\boldsymbol{a}_0$, and define, for all $j \geq 0$,

$$\boldsymbol{a}_{j+1} = S\boldsymbol{a}_j.$$

The sequence $\boldsymbol{a}_0$ can be viewed as a coarse approximation of a function, on the integer grid; the sequences $\boldsymbol{a}_j$ then give successively finer approximations of the function on grids with spacing 2^{-j}. We are interested in the case when this process converges to a smooth function as j increases. A subdivision scheme is *interpolating* if $s_{2l} = \delta_{l,0}$, implying $a_{j+1,2k} = a_{j,k}$ for all j, k; in this case the $a_{j,k}$ are interpreted as function values of $f, a_{j,k} = f(t_{j,k}) = f(2^{-j}k)$.

The *order* of a subdivision scheme S is the largest degree for which it leaves the corresponding space of monic polynomials invariant. More precisely, S is of order $\mathcal{P}$ if $\mathcal{P}$ is the largest integer such that for all p-degree monic polynomials P with $0 \leq p < \mathcal{P}$, a p-degree monic polynomial Q exists so that $SP(\boldsymbol{k}) = Q(\boldsymbol{k}/2)$. If S is interpolating, then $SP(\boldsymbol{k}) = P(\boldsymbol{k}/2)$. We always assume that $\mathcal{P}$ is at least one so that $S\mathbf{1} = \mathbf{1}$. The *derived* subdivision schemes are defined as

$$S^{[0]} = S, \qquad S^{[p]} = 2\Delta S^{[p-1]}\Delta^{-1}, \qquad p > 0.$$

Note that $S^{[p]}$ is well-defined as long as $S^{[p-1]}$ has at least order one, and that the order of $S^{[p]}$ is one less than the order of $S^{[p-1]}$. Thus $S^{[p]}$ is defined for $p \leq \mathcal{P}$. Also note that

$$S^{[p]} D_j = D_{j+1} S^{[p-1]} \quad \text{and} \quad S^{[p]} D_j^p = D_{j+1}^p S.$$

A special example is the midpoint interpolating subdivision scheme S_2 where $(S_2\boldsymbol{x})_{2k+1} = (x_k + x_{k+1})/2$. This scheme is used in Figure 1, has order $\mathcal{P} = 2$ and yields piecewise linear limit functions. We are now ready to state the main results.

3. Summary of Main Results

In this section we give a summary of the main theorems of this paper. This will help the reader understand the structure of the remainder of the paper. These theorems are given without proof and also typically are not necessarily the most general possible. This is because in their most general form, the statement of the theorem becomes much more technical and harder to read. For each less general theorem stated here we refer to the more general form and its proof later in the text. The less general theorems here typically omit any polynomial factors in the estimates and corresponding logarithmic factors in the regularity estimates. Therefore they do not necessarily provide the sharpest bounds on the fractional regularity.

Before we start with the statements of our results, we recall some technical preliminaries that will be used extensively, and that also set some of the notations used below.

3.1. **Technical preliminaries.** The first proposition states some basic estimates about subdivision that will be used later; for the sake of completeness we include their short proofs.

Proposition 3.1. *For a subdivision scheme S of order $\mathcal{P} \geq 1$ we have the estimates*

$$|(S\boldsymbol{x})_k| \leq C \max_{\ell \in I_k} |x_\ell|, \tag{3.1}$$

$$\max_{\ell \in I_k} |x_\ell - (S\boldsymbol{x})_k| \leq C \max_{\ell \in I_k} |(\Delta \boldsymbol{x})_\ell| \leq C \left|\Delta \boldsymbol{x}\right|_\infty = C 2^{-j} \left|D_j \boldsymbol{x}\right|_\infty , \tag{3.2}$$

Proof. The first estimate is given by the definition of S,

$$|(S\boldsymbol{x})_k| \leq \sum_{\ell \in I_k} |s_{k-2\ell}||x_\ell| \leq \max_{\ell \in I_k} |x_\ell| \sum_{\ell \in I_k} |s_{k-2\ell}|.$$

The second estimate follows from this first one, and from $S\mathbf{1} = \mathbf{1}$ since $\mathcal{P} \geq 1$,

$$\begin{aligned} \max_{\ell_1 \in I_k} |x_{\ell_1} - (S\boldsymbol{x})_k| &= \max_{\ell_1 \in I_k} |(S(x_{\ell_1}\mathbf{1} - \boldsymbol{x}))_k| \leq C \max_{\ell_1, \ell_2 \in I_k} |x_{\ell_1} - x_{\ell_2}| \\ &= C \max_{\substack{\ell_1, \ell_2 \in I_k \\ \ell_1 \leq \ell_2}} \left| \sum_{q=\ell_1}^{\ell_2 - 1} (\Delta \boldsymbol{x})_q \right| \leq CB \max_{\ell \in I_k} |(\Delta \boldsymbol{x})_\ell|. \end{aligned}$$

□

The estimate (3.1) states that S is a bounded operator when restricted to ℓ^∞, i.e.

$$|S|_\infty := \sup_{\boldsymbol{x} \in \ell^\infty, |\boldsymbol{x}|_\infty \leq 1} |S\boldsymbol{x}|_\infty < \infty.$$

Next, we introduce the function spaces we will be working with. Let $C^0(I)$ be the continuous and bounded functions defined on a (possibly unbounded) interval $I \subseteq \mathbb{R}$. Moreover, for p a positive integer, let $C^p(I)$ be constituted by the functions in $C^0(I)$ with a p-th derivative that is continuous and bounded on I. Our notations for fractional regularity is as follows. For $f \in C^0(I)$, let

$$\Omega(r, f) = \sup_{t_0, t_1 \in I} \frac{|f(t_0) - f(t_1)|}{|t_0 - t_1|^r}.$$

For $p \in \mathbb{N}$ and $0 < r < 1$ we define the class $C^{p+r}(I)$ as the set of functions $f \in C^p(I)$ for which $\Omega(r, f^{(p)})$ is bounded. Similarly, we use the notation $f \in \mathrm{Lip}^\alpha(I)$, with $\alpha = p + r$, $p \in \mathbb{N}$, $0 < r \leq 1$, when $f \in C^p(I)$ and $\Omega(r, f^{(p)})$ is bounded. For $\alpha \notin \mathbb{N}$, the spaces $\mathrm{Lip}^\alpha(I)$ and $C^\alpha(I)$ coincide; for $\alpha \in \mathbb{N}$, however, $C^\alpha(I) \not\subseteq \mathrm{Lip}^\alpha(I)$. Finally, $C^{\alpha^-}(I)$ or $\mathrm{Lip}^{\alpha^-}(I)$ stands for

$$\bigcap_{\alpha' < \alpha} C^{\alpha'}(I) = \bigcap_{\alpha' < \alpha} \mathrm{Lip}^{\alpha'}(I).$$

We shall use the notation α^- in more general contexts as well. More precisely, if r is a real number, we shall use the notation r^- wherever we could insert in its place $r - \varepsilon$ with $\varepsilon > 0$ arbitrarily small. With some abuse of notation we adopt the conventions $r^- < r'$ if $r \leq r'$ and $r < r'^-$ if $r < r'$. It follows that $\min(r^-, r')$ equals r' if $r > r'$, and r^- if $r \leq r'$.

We will often use the version of Taylor's theorem that says that if $f \in \mathrm{Lip}^{(p+r)}(I)$ and $p \in \mathbb{N}$, $0 < r \leq 1$ we can write

$$f(x) = \sum_{k=0}^{p} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R(x),$$

where the rest term $R(x)$ is bounded by

$$|R(x)| \leq \frac{\Omega\big(r, f^{(p)}\big)}{p!} |x - x_0|^{p+r}, \qquad \forall x, x_0 \in I.$$

3.2. **Our results.** The first theorem we state here is well known from the literature (a more general version can be found in [1]); we include it here for comparison with the theorems that follow it.

Theorem 3.2. *Let S be a subdivision scheme of order $\mathcal{P} \geq 1$ and $S^{[p]}$ its p-th derived scheme, with $p \leq \mathcal{P}$. Assume there are positive real numbers C, μ such that*

$$\left| S^{[p]^j} \right|_\infty \leq C 2^{\mu j}, \qquad \forall j \geq 0.$$

Let $\{\boldsymbol{x}_j\}$ be a family of sequences built by subdivision and let $\varphi_j(t)$ be a piecewise linear function interpolating the points $(x_{j,k})$ at $t = k2^{-j}$ for all j, k. Set

$$P + \kappa := p - \mu, \qquad P \in \mathbb{N}, \qquad 0 < \kappa \leq 1.$$

If $P \geq 0$ and $|\boldsymbol{x}_0|_\infty < \infty$, then there exists a function $\varphi \in C^{(P+\kappa)^-}(\mathbb{R})$ such that $\varphi_j \to \varphi$ uniformly exponentially.

The best bound one can get from this theorem is obtained for that combination of p and μ where $p - \mu$ is maximal. Note that this maximum need not be reached at $p = \mathcal{P}$.

The next theorem concerns sequences $\boldsymbol{x}_j$ that are not formed exactly by subdivision, but that are close in the sense that the difference between $\boldsymbol{x}_j$ and $S\boldsymbol{x}_{j-1}$ goes to zero exponentially.

Theorem 3.3. *Let S be a subdivision scheme of order $\mathcal{P} \geq 1$ and $S^{[p]}$ its p-th derived scheme, with $p \leq \mathcal{P}$. Assume there are positive real numbers C, μ such that*

$$\left| S^{[p]^j} \right|_\infty \leq C 2^{\mu j}, \qquad \forall j \geq 0.$$

Let $\{\boldsymbol{x}_j\}$ be a family of sequences satisfying

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \leq C 2^{-\nu j}, \qquad j \geq 0,$$

for some real number ν and let $\varphi_j(t)$ be a piecewise linear function interpolating the points $(x_{j,k})$ at $t = k2^{-j}$ for all j, k. Set

$$P + \kappa := \min(p - \mu, \nu), \qquad P \in \mathbb{N}, \qquad 0 < \kappa \leq 1.$$

If $P \geq 0$ and $|\boldsymbol{x}_0|_\infty < \infty$, then there exists a function $\varphi \in C^{(P+\kappa)^-}(\mathbb{R})$ such that $\varphi_j \to \varphi$ uniformly exponentially.

This theorem says that the regularity of the limit function of a family of sequences approximately generated by subdivision is bounded both by the regularity of the subdivision scheme and the speed of the approximation. The more general form of this theorem and is proof is given in Theorem 4.4.

Note that this is similar to standard results linking smoothness of functions with the decay of their wavelet coefficients, where the wavelet coefficients at level j correspond to the difference between $\boldsymbol{x}_{j+1}$ and $S\boldsymbol{x}_j$; in the wavelet case the subdivision operator S is determined by the low-pass filter corresponding to the wavelet basis.

Next we show that if you apply a smooth, but possibly non-linear function F to a family of subdivision sequences, you get an approximate family of subdivision sequences where the speed of approximation depends on the regularity of F. This is important because this typically will happen to the coordinate functionals in a normal scheme.

Theorem 3.4. *Let S be a subdivision scheme of order $\mathcal{P}$ and let $\{\boldsymbol{x}_j\}$ be generated by S. Suppose that $F \in C^{M+1}(\mathbb{R})$ with $M \in \mathbb{N}$ and $1 \leq M < \mathcal{P}$; suppose also that $\left|\boldsymbol{x}_j^{[m]}\right|_\infty \leq C$ for $1 \leq m \leq M$ and for $j > 0$. Then*

$$|F(S\boldsymbol{x}_j) - SF(\boldsymbol{x}_j)|_\infty \leq C 2^{-j(M+1)}.$$

The fully general version of this theorem is given in Theorem 4.7

We next go into more detail on the construction of a normal multiresolution for a smooth curve Γ in the plane. Even though the normal multiresolution algorithm does not depend on any parameterization, to formulate the theorems it is convenient to parameterize Γ by one of the x- or y-coordinates. A piecewise C^1 curve can always be broken up into adjacent finite length pieces, possibly overlapping, that can be well parameterized by the x-coordinate or by the y-coordinate; by restricting ourselves to these different pieces separately, and interchanging the names of the two coordinates, we may thus assume, without loss of generality, that the curve Γ is parameterized by its x-coordinate, so that

$$\Gamma = \{(x, \gamma(x));\ \ x \in I\},$$

where γ is a smooth function and I is a bounded interval. In general, we assume that γ is at least $C^1(I)$, but occasionally consider the more general case where γ is Hölder continuous with exponent $\beta \leq 1$.

Having reduced Γ (at least locally) to the graph of a function $\gamma(x)$, we can rephrase the basic step in the construction of a normal multiresolution given in Figure 2. We start with a sequence $\boldsymbol{x}_j$ on level j and define $\boldsymbol{y}_j = \gamma(\boldsymbol{x}_j)$. Next we use an interpolating subdivision scheme S to compute the sequences $\boldsymbol{x}^*_{j+1} = S\boldsymbol{x}_j$ and $\boldsymbol{y}^*_{j+1} = S\boldsymbol{y}_j$. In general $\boldsymbol{y}^*_{j+1}$ is not equal to $\gamma(\boldsymbol{x}^*_{j+1})$, but as we will see they are close. Next we draw the line through $(x^*_{j+1,2k+1},\ y^*_{j+1,2k+1})$ that is perpendicular to the line connecting $(x_{j,k},\ y_{j,k})$ and $(x_{j,k+1},\ y_{j,k+1})$. This line and the piece of Γ between $(x_{j,k},\ y_{j,k})$ and $(x_{j,k+1},\ y_{j,k+1})$ have to intersect in at least one point. We choose one of the intersection points to be the new point $(x_{j+1,2k+1},\ y_{j+1,2k+1} = \gamma(x_{j+1,2k+1}))$[1]. Given that $\boldsymbol{y}_j$ is always $\gamma(\boldsymbol{x}_j)$, we focus our attention on the convergence of the $\boldsymbol{x}_j$ sequences. We will call a family of sequences $\{\boldsymbol{x}_j\}$ defined by the above procedure, a family of sequences generated by the (S, γ) normal scheme.

To have a proper parameterization, we need that all $\boldsymbol{x}_j$ sequences are increasing, i.e., $\Delta\boldsymbol{x}_j > 0$. In general there are very few subdivision schemes that always preserve increasing sequences. In our case, the $\boldsymbol{x}_j$ are obtained by a nonlinear perturbation of subdivision so the situation is even more complex. Fortunately, there are conditions on both the subdivision scheme and the initial sequence that guarantee that the $\boldsymbol{x}_j$ will be increasing. The following theorem introduces a non-uniformity measure $\mathcal{N}$ of a sequence which is the maximal ratio of the length of two neighboring intervals; it states that if the non-uniformity of the initial sequence is bounded and the subdivision scheme preserves this bound, the sequences $\boldsymbol{x}_j$ generated by the normal scheme will be increasing and converge exponentially.

Theorem 3.5. *Let S be an interpolating subdivision scheme. Let the non-uniformity $\mathcal{N}(\boldsymbol{x})$ be defined by*

$$\mathcal{N}(\boldsymbol{x}) := \sup_k \max\left(\frac{|(\Delta\boldsymbol{x})_k|}{|(\Delta\boldsymbol{x})_{k+1}|}, \frac{|(\Delta\boldsymbol{x})_{k+1}|}{|(\Delta\boldsymbol{x})_k|}\right). \tag{3.3}$$

Suppose there is an R such that for every strictly increasing $\boldsymbol{x}$ with $\mathcal{N}(\boldsymbol{x}) \leq R$, $S\boldsymbol{x}$ is strictly increasing as well, and satisfies $\mathcal{N}(S\boldsymbol{x}) \leq \mathcal{N}(\boldsymbol{x})$. Suppose $\boldsymbol{x}_0$ is strictly increasing, with sufficiently small $|\Delta\boldsymbol{x}_0|_\infty$ and $\mathcal{N}(\boldsymbol{x}_0) < R$. If $\gamma \in C^2(\mathbb{R})$, then $\boldsymbol{x}_j$ is strictly increasing for all j, with $\mathcal{N}(\boldsymbol{x}_j) \leq R$ for all j, and the $\boldsymbol{x}_j$ converge exponentially, i.e., there is a $\delta < 1$ so that

$$|\Delta\boldsymbol{x}_j|_\infty \leq \delta^j\, |\Delta\boldsymbol{x}_0|_\infty\,, \qquad \forall j.$$

If $S = S_2$ the same conclusions follow if γ is merely Lipschitz continuous, without the smallness assumptions on $|\Delta\boldsymbol{x}_0|_\infty$ and $\mathcal{N}(\boldsymbol{x}_0)$.

This theorem in its full generality, which is more explicit on how small $|\Delta\boldsymbol{x}_0|_\infty$ needs to be, will be proven in Theorem 5.7.

Combining the results of Theorem 3.4 and Theorem 3.5, we can prove that the normal approximation procedure defines a new, smooth parameterization of $\Gamma = \{(x(t), \gamma(x(t))); t \in [0, 1]\}$, where the smoothness of the reparameterization is governed by the smoothness of γ as well as the regularity of the subdivision scheme.

One of the important features of a normal multiresolution is the decay of the offsets in each of the normal directions. We will refer to these as wavelet coefficients $\boldsymbol{w}_j$ which are defined as

$$w_{j,k} = \sqrt{(x_{j+1,2k+1} - x^*_{j+1,2k+1})^2 + (y_{j+1,2k+1} - y^*_{j+1,2k+1})^2}.$$

The rate of convergence to 0 of the wavelet coefficients is then determined by the order $\mathcal{P}$ and regularity of S, and the smoothness of Γ. The next theorem states these results, proved in more generality as Theorem 6.3.

[1]Our results below do not depend on which of these points is selected. For definiteness we shall assume that there is a rule established which uniquely picks out one of the solutions, should there be many. For instance we could systematically pick the solution closest to the predicted point.

Theorem 3.6. *Let S be an interpolating subdivision scheme of order $\mathcal{P} \geq 1$ and $S^{[p]}$ its p-th derived scheme, with $p \leq \mathcal{P}$. Assume there are positive real numbers C, μ such that*

$$\left| S^{[p]^j} \right|_\infty \leq C 2^{\mu j}, \qquad \forall j \geq 0, \qquad \mu \leq p - 1.$$

Let $\{\boldsymbol{x}_j\}$ be a family of increasing sequences generated by the (S, γ) normal scheme, for which there is a $\delta < 1$ so that

$$|\Delta \boldsymbol{x}_j|_\infty \leq C \delta^j.$$

Let $\mathtt{x}_j(t)$ be a piecewise linear function interpolating the points $x_{j,k}$ at $t = k2^{-j} \in [0,1]$. If $\gamma \in C^\beta(\mathbb{R})$ with $\beta \geq 2$ then $\mathtt{x}_j(t)$ converges uniformly exponentially to $\mathtt{x}(t)$ and $\mathtt{x} \in C^{Q^-}([0,1])$, where $Q := \min(p - \mu, \beta)$.

In addition let $Q' := \min(p - \mu + 1, \beta, \mathcal{P})$. Then for all $\varepsilon > 0$ there is a constant C_ε for which the wavelet coefficients,

$$w_{j,k} = \sqrt{(x_{j+1,2k+1} - (S\boldsymbol{x}_j)_{2k+1})^2 + (\gamma(x_{j+1,2k+1}) - (S\gamma(\boldsymbol{x}_j))_{2k+1})^2},$$

satisfy

$$|\boldsymbol{w}_j|_\infty \leq C_\varepsilon 2^{-j(Q' - \varepsilon)},$$

Finally if $Q > 1$, let $Q'' = \min(Q - 1, 1)$. Then for sufficiently large j and arbitrary $\varepsilon > 0$, there is a constant C_ε such that

$$\mathcal{N}(\boldsymbol{x}_j) - 1 \leq C_\varepsilon 2^{-j(Q'' - \varepsilon)},$$

with $\mathcal{N}(\boldsymbol{x}_j)$ defined as in (3.3).

Finally, we look at the stability of normal multiresolution. In particular we estimate how errors or round-offs in the wavelet coefficients affect the "reconstruction" of Γ. For pairs of sequences, $\boldsymbol{v} = (\boldsymbol{x}, \boldsymbol{y}) \in X^2$, we use the norm

$$|\boldsymbol{v}|_{2,\infty} = \sup_k \sqrt{x_k^2 + y_k^2}$$

Theorem 3.7. *We make the same assumptions as in Theorem* 3.6 *with the added provision that if $p > 1$ we need $\mu < p - 1$. Let $\boldsymbol{v}_j$ be the vector valued sequences defined in Figure* 2 *and let $\tilde{\boldsymbol{v}}_j$ be the corresponding sequences obtained when the curve is reconstructed from $\tilde{\boldsymbol{v}}_0$ with the perturbed wavelet coefficients $\tilde{\boldsymbol{w}}_j$. Suppose*

$$|\boldsymbol{v}_0 - \tilde{\boldsymbol{v}}_0|_{2,\infty} \leq E_f, \qquad |\boldsymbol{w}_j - \tilde{\boldsymbol{w}}_j|_\infty \leq E_w 2^{-js}, \qquad s > 0.$$

Then there is a constant C independent of j, E_f and E_w such that for $j > 0$,

$$|\boldsymbol{v}_j - \tilde{\boldsymbol{v}}_j|_{2,\infty} \leq C(E_f + E_w).$$

The more general version is given in Theorem 6.4. In particular, as shown in the discussion at the end of this theorem, this makes it possible to threshold wavelet coefficients and still obtain a high quality reconstruction.

4. Perturbing a Linear Subdivision Scheme

Certain linear subdivision schemes produce sequences that converge to smooth functions, as shown in e.g. [5, 7, 1, 3, 4, 2]. For instance, starting from an arbitrary $\boldsymbol{x}_0$ in ℓ^∞, the 4-point (interpolating) subdivision scheme of [5, 7] produces sequences $\boldsymbol{x}_j$ in ℓ^∞ such that, for all j,

$$\sup_k |f(2^{-j}k) - x_{j,k}| \leq C 2^{-j},$$

where f is a function in C^{2^-} depending on $\boldsymbol{x}_0$; many other subdivision schemes have similar convergence properties. In this section, we shall consider sequences $\boldsymbol{x}_j$ that are "almost" produced by such a linear subdivision process, in the sense that the difference between $\boldsymbol{x}_{j+1}$ and $S\boldsymbol{x}_j$ is small, and decays exponentially in j as j grows (see (4.1) below). We shall see that such perturbations still converge to a continuous limit function; moreover, provided the rate of decay of the perturbation is sufficiently fast, the smoothness of the limit function is not affected by the perturbation.

4.1. **General Assumptions.** Let $\{\boldsymbol{x}_j\}$ be a family of sequences, and suppose that there is a stationary subdivision scheme S, and constants $\nu > 0$ and $a, \alpha \geq 0$ such that

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \leq a(j+1)^\alpha 2^{-\nu j}, \qquad j \geq 0. \tag{4.1}$$

The order of the (linear, stationary, local and bounded) subdivision scheme S will always be denoted by $\mathcal{P}$; we shall consider only S for which $\mathcal{P} \geq 1$. We shall be interested in ℓ^∞-bounds on $(S^{[q]})^j$. We have of course for $0 \leq q \leq \mathcal{P}$,

$$\left|S^{[q]^j}\right|_\infty \leq \left|S^{[q]}\right|_\infty^j;$$

often we can provide tighter bounds. If the spectral radius ϱ_q of $S^{[q]}$ in ℓ^∞ is strictly smaller than $\left|S^{[q]}\right|_\infty$, then it follows from the well-known identity $\log \varrho_q = \lim_{j\to\infty} \frac{1}{j} \log \left|S^{[q]^j}\right|_\infty$ that we also have

$$\left|S^{[q]^j}\right|_\infty \leq C\left[\varrho_q(1+\varepsilon)\right]^j,$$

where $\varepsilon > 0$ is arbitrary, and C depends on ε. Estimates of this type will be used extensively below. In what follows, we shall assume that we pick one particular p with $0 \leq p \leq \mathcal{P}$ and a corresponding real number $\mu \geq 0$, such that

$$\left|S^{[p]^j}\right|_\infty \leq c2^{\mu j}, \qquad \forall j \geq 0, \tag{4.2}$$

for some c independent of j; we shall derive all our other estimates form (4.2). By allowing ourselves the freedom to choose $p \neq \mathcal{P}$, some of the derived estimates may be tighter than if we picked $p = \mathcal{P}$. If (4.2) is satisfied for the pair (p, μ), then we shall say that μ is *p-suitable*. We will see that if μ is p-suitable, then the Hölder regularity of the limiting functions obtained by applying pure subdivision to arbitrary initial sequences is at least $p - \mu$, up to possible logarithmic factors in the estimates. In the perturbation case, both $p - \mu$ and ν play a role (see below). This importance of the quantities $p - \mu$ motivates the following definitions. For each $p \in \{1, \dots, \mathcal{P}\}$, we define

$$\sigma_p = \sup\{p - \mu;\ \mu \geq 0,\ \text{there exists } c > 0 \text{ such that (4.2) is satisfied}\}.$$

Clearly, all $\mu > p - \sigma_p$ are p-suitable; $p - \sigma_p$ itself may or may not be p-suitable, depending on p and S. We define the *smoothness* σ to be the maximum of these σ_p:

$$\sigma := \max\{\sigma_p;\ p = 1, \dots, \mathcal{P}\}.$$

The p at which this maximum is achieved is called the *optimal* p, and (occasionally) denoted by p_{opt}. If there are two different maximizing p_1, p_2 for which $\sigma_{p_1} = \sigma_{p_2} = \sigma$, but for one of them $p_i - \sigma$ is p_i-suitable, then we pick this p as the optimal one. We define the *smoothness* σ of the subdivision scheme to be the value of σ_p for the optimal p. In some theorem statements, it is useful to use the notation

$$\hat{\sigma} := \begin{cases} \sigma, & p_{\mathrm{opt}} - \sigma \text{ is } p_{\mathrm{opt}}\text{-suitable}, \\ \sigma^-, & \text{otherwise.} \end{cases}$$

4.2. **A useful estimate for geometric series.** For convenience we define the geometric series function

$$\mathcal{G}(a, n, \alpha) = \sum_{k=0}^{n-1} k^\alpha a^k, \tag{4.3}$$

(where by convention $0^0 = 1$.) We will use this function in the subsequent sections, together with some well-known facts that we summarize in

Proposition 4.1. *For $a, \alpha \geq 0$ and $n \in \mathbb{Z}^+$ the function in* (4.3) *satisfies*

$$\mathcal{G}(a,n,\alpha) \leq \begin{cases} C(a) n^\alpha a^n, & a > 1, \\ n^{\alpha+1}, & a = 1, \\ C(a,\alpha), & a < 1, \end{cases} \tag{4.4}$$

for some constants C. Moreover, $\mathcal{G}$ is increasing in all of its arguments; if $a < b$, there is a constant C independent of n such that

$$\mathcal{G}(a,n,\alpha_1) \leq C(a,b,\alpha_1,\alpha_2)\ \mathcal{G}(b,n,\alpha_2). \tag{4.5}$$

Finally, for $0 \leq n_1 < n_2$,

$$\sum_{k=n_1}^{n_2-1} k^\alpha a^k \leq (1+n_1)^\alpha a^{n_1}(1 + \mathcal{G}(a, n_2 - n_1, \alpha)). \tag{4.6}$$

4.3. **Estimates and Regularity Results.** We start by establishing some estimates in preparation for our main goal in this subsection: Theorem 4.4 below.

Theorem 4.2. *Let $\{\boldsymbol{x}_j\}$, α, a, ν, S, $\mathcal{P}$ be given as in Section* 4.1*; pick $p \in \{1, \dots, \mathcal{P}\}$ and let μ be p-suitable. Set*

$$\varrho = \max(p - \nu, \mu).$$

Then there is a constant C, independent of j, a and $\boldsymbol{x}_0^{[q]}$, such that

$$\begin{aligned} \left|\boldsymbol{x}_j^{[q]}\right|_\infty &\leq C\left(\left|\boldsymbol{x}_0^{[q]}\right|_\infty + a\right)\left[1 + j^{\eta_q} 2^{j(q-p+\varrho)}\right] \\ &\leq C\left(\left|\boldsymbol{x}_0^{[q]}\right|_\infty + a\right) \begin{cases} j^{\eta_q} 2^{j(q-p+\varrho)}, & \varrho \geq p - q, \\ 1, & \varrho < p - q, \end{cases} \qquad 0 \leq q \leq p, \end{aligned} \tag{4.7}$$

where

$$\eta_q = \begin{cases} 0, & \mu > p - \nu, \\ \alpha + 1, & \mu = p - \nu, \\ \alpha, & \mu < p - \nu, \end{cases} \quad + \quad \begin{cases} 1, & \varrho = p - q > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{4.8}$$

Proof. Let us start by defining the residual sequences

$$\boldsymbol{r}_{j+1} := \boldsymbol{x}_{j+1} - S\boldsymbol{x}_j, \qquad \boldsymbol{r}_0 = \boldsymbol{x}_0;$$

observe that

$$\boldsymbol{r}_{j+1}^{[p]} = \boldsymbol{x}_{j+1}^{[p]} - S^{[p]}\boldsymbol{x}_j^{[p]}.$$

By induction on the simple relationship

$$\left|\boldsymbol{r}_{j+1}^{[q+1]}\right|_\infty = 2^{j+1}\left|\Delta \boldsymbol{r}_{j+1}^{[q]}\right|_\infty \leq c2^{j+1}\left|\boldsymbol{r}_j^{[q]}\right|_\infty,$$

and (4.1), we get

$$\left|\boldsymbol{r}_j^{[p]}\right|_\infty \leq c2^{jp}\left|\boldsymbol{r}_j\right|_\infty \leq caj^\alpha 2^{-j(\nu-p)}, \qquad \forall j > 0. \tag{4.9}$$

Together (4.9) and (4.2) give us for $j \geq 0$,

$$\begin{aligned}
\left|\boldsymbol{x}_j^{[p]}\right|_\infty &= \left|S^{[p]}\boldsymbol{x}_{j-1}^{[p]} + \boldsymbol{r}_j^{[p]}\right|_\infty = \left|{S^{[p]}}^2\boldsymbol{x}_{j-2}^{[p]} + S^{[p]}\boldsymbol{r}_{j-1}^{[p]} + \boldsymbol{r}_j^{[p]}\right|_\infty \\
&= \left|\sum_{q=0}^{j} {S^{[p]}}^q \boldsymbol{r}_{j-q}^{[p]}\right|_\infty \leq \left|{S^{[p]}}^j\right|_\infty \left|\boldsymbol{r}_0^{[p]}\right|_\infty + ca\sum_{q=0}^{j-1}\left|{S^{[p]}}^q\right|_\infty (j-q)^\alpha 2^{-(j-q)(\nu-p)} \\
&\leq c2^{\mu j}\left|\boldsymbol{x}_0^{[p]}\right|_\infty + Ca\sum_{q=0}^{j-1} 2^{\mu q}(j-q)^\alpha 2^{-(j-q)(\nu-p)} = c2^{\mu j}\left|\boldsymbol{x}_0^{[p]}\right|_\infty + Ca2^{\mu j}\mathcal{G}(2^{p-\nu-\mu}, j+1, \alpha) \\
&\leq c2^{\mu j}\left|\boldsymbol{x}_0^{[p]}\right|_\infty + Ca\begin{cases}(j+1)^\alpha 2^{j(p-\nu)}, & \mu < p-\nu, \\ (j+1)^{\alpha+1}2^{\mu j}, & \mu = p-\nu, \\ 2^{\mu j}, & \mu > p-\nu,\end{cases} \leq C\left(\left|\boldsymbol{x}_0^{[p]}\right|_\infty + a\right)(1 + j^{\eta_p}2^{\varrho j}),
\end{aligned} \tag{4.10}$$

which agrees with (4.7, 4.8) when $q = p$. Suppose now that (4.7, 4.8) holds for some $q \leq p$. Induction will then yield the result if we can prove that this implies (4.7, 4.8) is true also for $q - 1 \geq 0$. To show this, we first fix an index $k =: k_{j+1}$, and construct a sequence of indices $\{k_s\}_{s=0}^{j}$ such that $k_s \in I_{k_{s+1}}$. Then

$$x_{j+1,k}^{[q-1]} = x_{0,k_0}^{[q-1]} + \sum_{s=0}^{j}\left(x_{s+1,k_{s+1}}^{[q-1]} - x_{s,k_s}^{[q-1]}\right)$$

and we can estimate

$$\left|\boldsymbol{x}_{j+1}^{[q-1]}\right|_\infty \leq \left|\boldsymbol{x}_0^{[q-1]}\right|_\infty + \sum_{s=0}^{j}\sup_k \max_{\ell\in I_k}\left|x_{s+1,k}^{[q-1]} - x_{s,\ell}^{[q-1]}\right|.$$

The desired result then follows from Lemma 4.3 (below) with $j_1 = 0$ and noting that $\left|\boldsymbol{x}_0^{[q]}\right|_\infty \leq 2\left|\boldsymbol{x}_0^{[q-1]}\right|_\infty$. ☐

Lemma 4.3. *With the assumptions and notation of Theorem* 4.2*, if* (4.7, 4.8) *holds for some* $q \leq p$ *then for* $0 \leq j_1 < j_2$,

$$\sum_{s=j_1}^{j_2-1}\sup_k\max_{\ell\in I_k}\left|x_{s+1,k}^{[q-1]} - x_{s,\ell}^{[q-1]}\right| \leq$$
$$C\left(\left|\boldsymbol{x}_0^{[q]}\right|_\infty + a\right)\left[2^{-j_1} + (1+j_1)^{\eta_q}2^{j_1(q-p-1+\varrho)}\left(1 + (j_2-j_1)^{\eta_{q-1}}2^{(j_2-j_1)(q-p-1+\varrho)}\right)\right].$$

Proof. Since $q - 1 < \mathcal{P}$ the order of $S^{[q-1]}$ is at least one, and we can use (3.2) in Proposition 3.1. Together with (4.9) and the hypothesis that (4.7, 4.8) is true for q, we then get

$$\begin{aligned}
\sum_{s=j_1}^{j_2-1}\sup_k\max_{\ell\in I_k}\left|x_{s+1,k}^{[q-1]} - x_{s,\ell}^{[q-1]}\right| &\leq \sum_{s=j_1}^{j_2-1}\sup_k\max_{\ell\in I_k}\left|\left(S^{[q-1]}\boldsymbol{x}_s^{[q-1]}\right)_k - x_{s,\ell}^{[q-1]}\right| + \sum_{s=j_1}^{j_2-1}\left|\boldsymbol{r}_{s+1}^{[q-1]}\right|_\infty \\
&\leq c\sum_{s=j_1}^{j_2-1}2^{-s}\left|\boldsymbol{x}_s^{[q]}\right|_\infty + ca\sum_{s=j_1}^{j_2-1}s^\alpha(2^{q-\nu-1})^s \\
&\leq c\left(\left|\boldsymbol{x}_0^{[q]}\right|_\infty + a\right)\sum_{s=j_1}^{j_2-1}2^{-s}\left[1 + s^{\eta_q}2^{(q-p+\varrho)s}\right] + ca\sum_{s=j_1}^{j_2-1}s^\alpha 2^{(q-p-1+\tilde{\mu})s},
\end{aligned}$$

where we set $\tilde{\mu} := p - \nu$ so that $\varrho = \max(\mu, \tilde{\mu})$. We now apply (4.6) in Proposition 4.1,

$$\sum_{s=j_1}^{j_2-1} \sup_k \max_{\ell \in I_k} \left| x_{s+1,k}^{[q-1]} - x_{s,\ell}^{[q-1]} \right| \leq$$
$$C\left(\left| \boldsymbol{x}_0^{[q]} \right|_\infty + a \right) \Big[2^{-j_1}(1 + \mathcal{G}(1/2, j_2 - j_1, 0))$$
$$+ (1+j_1)^{\eta_q} 2^{j_1(q-p-1+\varrho)} (1 + \mathcal{G}(2^{q-p-1+\varrho}, j_2 - j_1, \eta_q)$$
$$+ (1+j_1)^{\alpha} 2^{j_1(q-p-1+\tilde{\mu})} (1 + \mathcal{G}(2^{q-p-1+\tilde{\mu}}, j_2 - j_1, \alpha) \Big].$$

Now, if $\mu \leq \tilde{\mu}$, then $\varrho = \tilde{\mu}$ and $\eta_q \geq \alpha$, and by Proposition 4.1,

$$\sum_{s=j_1}^{j_2-1} \sup_k \max_{\ell \in I_k} \left| x_{s+1,k}^{[q-1]} - x_{s,\ell}^{[q-1]} \right| \leq C\left(\left| \boldsymbol{x}_0^{[q]} \right|_\infty + a \right) \Big[2^{-j_1} + (1+j_1)^{\eta_q} 2^{j_1(q-p-1+\varrho)} (1 + \mathcal{G}(2^{q-p-1+\varrho}, j_2 - j_1, \eta_q)) \Big].$$

Similarly, if $\mu > \tilde{\mu}$, we have that $\varrho = \mu$ and by (4.5) in Proposition 4.1 we get the same result. Finally, for $j \geq 0$,

$$\mathcal{G}\left(2^{q-p-1+\varrho}, j, \eta_q\right) \leq c \begin{cases} j^{\eta_q} 2^{j(q-p-1+\varrho)}, & \varrho > p - q + 1, \\ j^{\eta_q+1}, & \varrho = p - q + 1, \\ 1, & \varrho < p - q + 1, \end{cases} \leq c\Big(1 + j^{\eta_{q-1}} 2^{j(q-p-1+\varrho)}\Big).$$

This concludes the proof. □

The estimate in Theorem 4.2 can now be used to prove a theorem about existence and regularity of the subdivision limit function.

Theorem 4.4. *Let $\{\boldsymbol{x}_j\}$, α, a, ν, S, $\mathcal{P}$, p and μ be given as in Section* 4.1*. Suppose $x_{j,k}$ is defined precisely for those j, k such that $t_{j,k} = k2^{-j} \in I$, where I is a (possibly infinite) interval. Let $\varphi_j(t)$ be a piecewise linear function interpolating the points $(x_{j,k})$ at $(t_{j,k})$. Set*

$$\varrho = \max(p - \nu, \mu), \qquad Q = p - \varrho = P + \kappa, \qquad P \in \mathbb{N}, \qquad 0 < \kappa \leq 1. \tag{4.11}$$

If $Q > 0$ and $|\boldsymbol{x}_0|_\infty < \infty$, then there exists a function $\varphi \in C^{Q^-}(I)$ such that $\varphi_j \to \varphi$ uniformly exponentially, and $\varphi^{(P)}$ satisfies the Hölder inequality

$$|\varphi^{(P)}(t + \Delta t) - \varphi^{(P)}(t)| \leq c|\Delta t|^\kappa (1 + |\log|\Delta t||)^\eta, \qquad \forall t,\ t + \Delta t \in I, \tag{4.12}$$

where

$$\eta = \begin{cases} 0, & \mu > p - \nu, \\ \alpha + 1, & \mu = p - \nu, \\ \alpha, & \mu < p - \nu, \end{cases} + \begin{cases} 1, & \varrho \in \mathbb{Z}^+, \\ 0, & \text{otherwise}. \end{cases} \tag{4.13}$$

Proof. Let $\varphi_j^{[q]}(t)$ be the piecewise linear function defined on I that interpolates the points $\left(x_{j,k}^{[q]}\right)$ at $(t_{j,k})$ for $0 \leq q \leq P$ (extrapolated as a constant at the boundaries when necessary). By Theorem 4.2 these functions are bounded uniformly for all j. Moreover, for $j_1 < j_2$,

$$\begin{aligned} \left| \varphi_{j_2}^{[q]} - \varphi_{j_1}^{[q]} \right|_\infty &\leq \sum_{s=j_1}^{j_2-1} \sup_t \left| \varphi_{s+1}^{[q]}(t) - \varphi_s^{[q]}(t) \right| = \sum_{s=j_1}^{j_2-1} \sup_k \left| \varphi_{s+1}^{[q]}\left(k2^{-(s+1)}\right) - \varphi_s^{[q]}\left(k2^{-(s+1)}\right) \right| \\ &= \sum_{s=j_1}^{j_2-1} \sup_k \left| x_{s+1,k}^{[q]} - \frac{1}{2}\left(x_{s,\lfloor k/2 \rfloor}^{[q]} + x_{s,\lceil k/2 \rceil}^{[q]} \right) \right| \leq \sum_{s=j_1}^{j_2-1} \sup_k \max_{\ell \in I_k} \left| x_{s+1,k}^{[q]} - x_{s,\ell}^{[q]} \right|. \end{aligned}$$

Since $q \le P < p$ we can use Lemma 4.3 to estimate this,

$$\begin{aligned}\left|\varphi_{j_2}^{[q]} - \varphi_{j_1}^{[q]}\right|_\infty &\le C\left(\left|\boldsymbol{x}_0^{[q+1]}\right|_\infty + a\right)\left[2^{-j_1} + (1+j_1)^\eta 2^{-j_1(P+\kappa-q)}\left(1 + (j_2-j_1)^{\eta_q} 2^{-(j_2-j_1)(P+\kappa-q)}\right)\right] \\ &\le C\left(2^{q+1}\left|\boldsymbol{x}_0\right|_\infty + a\right)\left[2^{-j_1} + (1+j_1)^\eta 2^{-j_1(P+\kappa-q)}\left(1 + (j_2-j_1)^{\eta_q} 2^{-(j_2-j_1)(P+\kappa-q)}\right)\right], \qquad (4.14)\end{aligned}$$

which tends to 0 when $j_1, j_2 \to \infty$. Hence $\{\varphi_j^{[q]}\}$ is a Cauchy sequence, and the limit $\varphi^{[q]} \in C^0(I)$ exists for $0 \le q \le P$. Next, after letting $j_2 \to \infty$ in (4.14) we get

$$\left|\varphi^{[q]} - \varphi_j^{[q]}\right|_\infty \le c j^\eta 2^{-j(P+\kappa-q)}, \qquad j > 0, \quad 0 \le q \le P.$$

This shows that the convergence rate is uniformly exponential. Then, taking Δt such that $2^{-j} \le |\Delta t| < 2^{-j+1}$ with $j > 1$ and using Theorem 4.2 again,

$$\begin{aligned}\left|\varphi^{[P]}(t+\Delta t) - \varphi^{[P]}(t)\right|_\infty &\le \left|\varphi^{[P]}(t+\Delta t) - \varphi_j^{[P]}(t+\Delta t)\right|_\infty + \left|\varphi_j^{[P]}(t+\Delta t) - \varphi_j^{[P]}(t)\right|_\infty \\ &\quad + \left|\varphi_j^{[P]}(t) - \varphi^{[P]}(t)\right|_\infty \\ &\le c j^\eta 2^{-j\kappa} + \left|\varphi_j^{[P]}(t+\Delta t) - \varphi_j^{[P]}(t)\right|_\infty \\ &\le c j^\eta 2^{-j\kappa} + 2^{-j}\left|\boldsymbol{x}_j^{[P+1]}\right|_\infty \le c j^\eta 2^{-j\kappa} \\ &\le c|\Delta t|^\kappa |\log|\Delta t||^\eta.\end{aligned}$$

This shows (4.12) for $|\Delta t| < 1/2$. Finally, for $|\Delta t| \ge 1/2$, (4.12) holds by the boundedness of $\varphi^{[P]}$. It remains to note that the functions $\varphi^{[p]}$ are related to φ as $p!\varphi^{[p]}(t) = d^p\varphi(t)/dt^p$ [1, 6]. □

In Theorems 4.2 and 4.4 we let μ take an arbitrary p-suitable value. The results are of course sharper for lower μ. If we take $p = p_{\text{opt}}$, and if $p - \sigma$ is itself p-suitable, then we can simply replace μ by $p - \sigma$ everywhere. If $p - \sigma$ is not p-suitable, then we can take $\mu = p - \sigma + \varepsilon$ where ε is arbitrarily small. On making these substitutions, we obtain the following corollary from Theorem 4.4.

Corollary 4.5. *The expression* (4.11) *can be replaced by*

$$Q = \min(\hat{\sigma}, \nu), \qquad Q =: P + \kappa, \qquad P \in \mathbb{N}, \qquad 0 < \kappa \le 1. \qquad (4.15)$$

We then have $\varphi \in C^{Q^-}(\mathbb{R})$ *and* (4.12) *holds with this* κ *and the following* η

$$\eta = \begin{cases} 0, & \sigma < \nu, \\ \alpha + 1, & \sigma = \nu, \\ \alpha, & \sigma > \nu, \end{cases} \quad + \quad \begin{cases} 1, & Q \in \{1, \dots, p-1\},\ p \ge 2, \\ 0, & \text{otherwise.} \end{cases} \qquad (4.16)$$

Remark: With the notation introduced in Theorem 4.4 and Corollary 4.5, we can rewrite (4.7, 4.8) as

$$\left|\boldsymbol{x}_j^{[q]}\right|_\infty \le \text{const}\left(\left|\boldsymbol{x}_0^{[q]}\right|_\infty + a\right)\begin{cases} 1, & 0 \le q \le P, \\ j^\eta 2^{j(1-\kappa)}, & q = P + 1. \end{cases} \qquad (4.17)$$

This estimate will be used below, in this form.

4.4. **Approximate Commutation.** In this section we explore the properties of the family of sequences $F(\boldsymbol{x}_j)$, where F is a smooth function, and the $\boldsymbol{x}_j$ are produced by subdivision, $\boldsymbol{x}_{j+1} = S\boldsymbol{x}_j$. We shall see that the $F(\boldsymbol{x}_j)$ constitute a family that is "almost" produced by subdivision. More precisely, we shall show that they satisfy an estimate of type (4.1), i.e.

$$|F(\boldsymbol{x}_{j+1}) - SF(\boldsymbol{x}_j)|_\infty = |F(S\boldsymbol{x}_j) - SF(\boldsymbol{x}_j)|_\infty \le c\, j^\kappa 2^{-j\nu}, \qquad (4.18)$$

with κ and ν to be determined below. It then follows that all the estimates of Section 4 can be applied to the $F(\boldsymbol{x}_j)$, or to any family of sequences $(\boldsymbol{y}_j)$ for which $\left|\boldsymbol{y}_{j+1} - S\boldsymbol{y}_j\right|_\infty$ is bounded by (4.18). This fact shall be exploited later. We start by proving a lemma for a fixed level.

Lemma 4.6. *Let S be an interpolating, stationary, local, linear and bounded subdivision scheme of order $\mathcal{P}$. Suppose*

$$F \in C^{M+r}(\mathbb{R}), \qquad M \in \mathbb{N}, \qquad 0 < r \leq 1. \tag{4.19}$$

If $M \leq 1$ and $\mathcal{P} \geq 1$,

$$\left|\left(F(S\boldsymbol{x}) - SF(\boldsymbol{x})\right)_K\right| \leq \; C \max_{k \in I_K} |(\Delta \boldsymbol{x})_k|^{M+r} \leq C\, |\Delta \boldsymbol{x}|_\infty^{M+r} \tag{4.20}$$

for all K, where C depends only on F and on S. If $M > 1$, $\mathcal{P} > M$, and

$$|\Delta^m \boldsymbol{x}|_\infty \leq 1, \qquad 1 \leq m \leq M, \tag{4.21}$$

then

$$|F(S\boldsymbol{x}) - SF(\boldsymbol{x})|_\infty \leq \; C \max\left(|\Delta \boldsymbol{x}|_\infty^{M+r},\ \left|\Delta^M \boldsymbol{x}\right|_\infty^2,\ A\ \left|\Delta^M \boldsymbol{x}\right|_\infty,\ A^{M+1}\right), \tag{4.22}$$

where

$$A := \max_{1 \leq m \leq M-1} |\Delta^m \boldsymbol{x}|_\infty^{1/m}.$$

Proof. Let $\boldsymbol{y} = F(S\boldsymbol{x}) - SF(\boldsymbol{x})$. Since $F \in C^{M+r}(\mathbb{R})$ we can Taylor expand $F(\boldsymbol{x})$ around $(S\boldsymbol{x})_K$, with K fixed. By also using the fact that $S\mathbf{1} = \mathbf{1}$, we get

$$\begin{aligned} y_K &= \left(S \sum_{n=1}^{M} (\boldsymbol{x} - (S\boldsymbol{x})_K)^n \frac{F^{(n)}((S\boldsymbol{x})_K)}{n!}\right)_K + (SR(\boldsymbol{x}))_K \\ &= \sum_{n=2}^{M} (S(\boldsymbol{x} - (S\boldsymbol{x})_K)^n)_K \frac{F^{(n)}((S\boldsymbol{x})_K)}{n!} + (SR(\boldsymbol{x}))_K, \end{aligned} \tag{4.23}$$

and the rest term, R, satisfies the estimate $|R(x)| \leq C|x - (S\boldsymbol{x})_K|^{M+r}$. (Note, (4.23) is true also for $M \leq 1$, with no contribution from the sum.) Using (3.1, 3.2) in Proposition 3.1 we get

$$|(SR(\boldsymbol{x}))_K| \leq C \max_{k \in I_K} |R(x_k)| \leq C \max_{k \in I_K} |x_k - (S\boldsymbol{x})_K|^{M+r} \leq C\ \max_{k \in I_K} |(\Delta \boldsymbol{x})_k|^{M+r} \leq C\ |\Delta \boldsymbol{x}|_\infty^{M+r}, \tag{4.24}$$

which shows (4.20), the case when $M \leq 1$.

For the case $M > 1$ we start by making a special discrete Taylor expansion around index K,

$$\boldsymbol{x} = \sum_{m=0}^{M-1} \frac{(\Delta^m S\boldsymbol{x})_K}{m!} Q_m(2\boldsymbol{k} - K) + \boldsymbol{r}, \tag{4.25}$$

where

$$Q_0 = 1, \qquad Q_m(x) = x(x-1)\cdots(x-m+1), \qquad m > 0, \tag{4.26}$$

and $\boldsymbol{r}$ is defined as the residual of the expression. (We keep in mind that $\boldsymbol{r}$ also depends on K, but for simplicity we do not make it explicit in the notation.) Before continuing, we derive an estimate for the residual $\boldsymbol{r}$. The polynomial Q_m in (4.25) corresponds to x^m in the Taylor expansion for the continuous case. We use it here since the effect of applying Δ to Q_m mimics the behavior of the continuous differentiation operator in the sense that $\Delta Q_m(\boldsymbol{k} - K) = mQ_{m-1}(\boldsymbol{k} - K)$, which is an easy consequence of (4.26) and (2.1). Induction on this relation gives

$$\Delta^n Q_m(\boldsymbol{k} - K) = \begin{cases} \frac{m!}{(m-n-1)!} Q_{m-n}(\boldsymbol{k} - K), & n < m, \\ m!\mathbf{1}, & n = m, \\ 0, & n > m. \end{cases}$$

Then, since $Q_m(2\boldsymbol{k}-K)$ is a linear combination of $\{Q_{m'}(\boldsymbol{k})\}_{m'\le m}$, clearly $\Delta^M\boldsymbol{x}=\Delta^M\boldsymbol{r}$. For $0\le n\le M-1$, we use the fact that S is interpolating of order $\mathcal{P}>M-1$, and compute

$$\begin{aligned}\Delta^n S\boldsymbol{x} &= \sum_{m=0}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{m!}\Delta^n SQ_m(2\boldsymbol{k}-K)+\Delta^n S\boldsymbol{r}\\ &= \sum_{m=0}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{m!}\Delta^n Q_m(\boldsymbol{k}-K)+\Delta^n S\boldsymbol{r}\\ &= (\Delta^n S\boldsymbol{x})_K\,\mathbf{1}+\sum_{m=n+1}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{(m-n-1)!}Q_{m-n}(\boldsymbol{k}-K)+\Delta^n S\boldsymbol{r}.\end{aligned}$$

By taking the K-th element of this sequence we conclude that

$$\left(S^{[n]}\Delta^n\boldsymbol{r}\right)_K=2^n\,(\Delta^n S\boldsymbol{r})_K=0,\qquad n=0,\dots,M-1.$$

Then, using (3.2) in Proposition 3.1 we have

$$\max_{k\in I_K}\left|(\Delta^{M-1}\boldsymbol{r})_k\right|=\max_{k\in I_K}\left|(\Delta^{M-1}\boldsymbol{r})_k-\left(S^{[M-1]}\Delta^{M-1}\boldsymbol{r}\right)_K\right|\le C\max_{k\in I_K}\left|(\Delta^M\boldsymbol{r})_k\right|.$$

By induction we get the estimate of the residual around the index K,

$$\max_{k\in I_K}|r_k|\le C\max_{k\in I_K}\left|(\Delta^M\boldsymbol{r})_k\right|=C\max_{k\in I_K}\left|(\Delta^M\boldsymbol{x})_k\right|\le C\left|\Delta^M\boldsymbol{x}\right|_\infty. \tag{4.27}$$

Going back to (4.23) we take n such that $2\le n\le M$ and consider

$$\begin{aligned}S(\boldsymbol{x}-(S\boldsymbol{x})_K)^n &= S\left(\sum_{m=1}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{m!}Q_m(2\boldsymbol{k}-K)+\boldsymbol{r}\right)^n\\ &= S\boldsymbol{r}^n+S\left(\sum_{m=1}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{m!}Q_m(2\boldsymbol{k}-K)\right)^n\\ &\quad+\sum_{s=1}^{n-1}\binom{n}{s}S\left[\boldsymbol{r}^s\left(\sum_{m=1}^{M-1}\frac{(\Delta^m S\boldsymbol{x})_K}{m!}Q_m(2\boldsymbol{k}-K)\right)^{n-s}\right]\\ &=: \boldsymbol{T}^1+\boldsymbol{T}^2+\boldsymbol{T}^3.\end{aligned}$$

For the first term $\boldsymbol{T}^1$ we get from (4.27),

$$|T^1_K|=|(S\boldsymbol{r}^n)_K|\le C\max_{k\in I_K}|r_k|^n\le C\left|\Delta^M\boldsymbol{x}\right|_\infty^n\le C\left|\Delta^M\boldsymbol{x}\right|_\infty^2. \tag{4.28}$$

Recalling that $S^{[m]}$ is bounded for $m\le M-1$ we furthermore have

$$|(\Delta^m S\boldsymbol{x})_K|=|2^{-m}(S^{[m]}\Delta^m\boldsymbol{x})_K|\le c\,|\Delta^m\boldsymbol{x}|_\infty, \tag{4.29}$$

and we get for $\boldsymbol{T}^3$,

$$\begin{aligned}|T^3_K| &\le C\sum_{s=1}^{n-1}\max_{k\in I_K}|r^s_k|\left(\sum_{m=1}^{M-1}|\Delta^m\boldsymbol{x}|_\infty\,|Q_m(2k-K)|\right)^{n-s}\\ &\le C\left|\Delta^M\boldsymbol{x}\right|_\infty\sum_{s=1}^{n-1}\max_{1\le m\le M-1}|\Delta^m\boldsymbol{x}|_\infty^{\frac{n-s}{m}}\le C\,A\,\left|\Delta^M\boldsymbol{x}\right|_\infty.\end{aligned} \tag{4.30}$$

Finally, for the second term $\boldsymbol{T}^2$, let $\beta = (\beta_1, \dots, \beta_N)$ denote a multi-index. Then, since S is interpolating of order $\mathcal{P} > M$ and $Q_m(0) = 0$ for $m \geq 1$,

$$\begin{aligned} T^2_K &= \left(S \sum_{|\beta|=n} \frac{n!}{\beta!} \prod_{m=1}^{M-1} \left(\frac{(\Delta^m S\boldsymbol{x})_K}{m!} \right)^{\beta_m} Q_m(2\boldsymbol{k}-K)^{\beta_m} \right)_K \\ &= \left(S \sum_{\substack{|\beta|=n \\ \sum_{m=1}^{M-1} m\beta_m > M}} \frac{n!}{\beta!} \prod_{m=1}^{M-1} \left(\frac{(\Delta^m S\boldsymbol{x})_K}{m!} \right)^{\beta_m} Q_m(2\boldsymbol{k}-K)^{\beta_m} \right)_K . \end{aligned}$$

By (4.29) we then have

$$\begin{aligned} |T^2_K| &\leq C \max_{k \in I_K} \sum_{\substack{|\beta|=n \\ \sum_{m=1}^{M-1} m\beta_m > M}} \prod_{m=1}^{M-1} |\Delta^m \boldsymbol{x}|_\infty^{\beta_m} Q_m(2k-K)^{\beta_m} \\ &\leq C \sum_{\substack{|\beta|=n \\ \sum_{m=1}^{M-1} m\beta_m > M}} \prod_{m=1}^{M-1} \left(|\Delta^m \boldsymbol{x}|_\infty^{1/m} \right)^{m\beta_m} \leq C \sum_{\substack{|\beta|=n \\ \sum_{m=1}^{M-1} m\beta_m > M}} \prod_{m=1}^{M-1} A^{m\beta_m} \leq C\, A^{M+1}. \end{aligned} \tag{4.31}$$

In conclusion, (4.22) follows from (4.24) and (4.23) together with the bounds (4.28, 4.30, 4.31). ☐

Theorem 4.7. *Let S be an interpolating, stationary, local, linear and bounded subdivision scheme of order $\mathcal{P}$ and let $\{\boldsymbol{x}_j\}$ be generated by S. Suppose*

$$F \in C^{M+r}(\mathbb{R}), \qquad M \in \mathbb{N}, \qquad 0 < r \leq 1.$$

If $M \leq 1$, $\mathcal{P} \geq 1$ and $\left|\boldsymbol{x}_j^{[1]}\right|_\infty \leq j^\alpha 2^{js}$, then

$$|F(S\boldsymbol{x}_j) - SF(\boldsymbol{x}_j)|_\infty \leq \text{const}\; j^{\alpha(M+r)} 2^{-j(M+r)(1-s)}. \tag{4.32}$$

If $M > 1$, $\mathcal{P} > M$ and

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \leq \text{const} \begin{cases} 1, & 1 \leq m < M, \\ j^\alpha 2^{js}, & m = M, \end{cases} \qquad \alpha \geq 0, \qquad 0 \leq s < M, \tag{4.33}$$

for $j > 0$, then

$$|F(S\boldsymbol{x}_j) - SF(\boldsymbol{x}_j)|_\infty \leq \text{const} \begin{cases} 2^{-j(M+r)}, & 0 \leq s < 1-r < M-1, \\ j^\alpha 2^{-j(M+1-s)}, & 0 \leq 1-r \leq s < M-1, \\ j^{2\alpha} 2^{-j(2M-2s)}, & 0 < M-1 \leq s < M. \end{cases} \tag{4.34}$$

Proof. The first result, (4.32), follows directly from Lemma 4.6, since $|\Delta\boldsymbol{x}|_\infty = 2^{-j}\left|\boldsymbol{x}_j^{[1]}\right|_\infty$.

For the second part we note that for $1 \leq m \leq M-1$, we have $|\Delta^m \boldsymbol{x}_j|_\infty \leq c\, 2^{-jm} \leq 1$ for sufficiently large j, and

$$A = \max_{1 \leq m \leq M-1} |\Delta^m \boldsymbol{x}_j|_\infty^{1/m} \leq \text{const} \max_{1 \leq m \leq M-1} 2^{-j} = \text{const}\, 2^{-j}.$$

Therefore,

$$\begin{aligned} &\max\left(|\Delta\boldsymbol{x}_j|_\infty^{M+r},\ \left|\Delta^M \boldsymbol{x}_j\right|_\infty^2,\ A\left|\Delta^M \boldsymbol{x}_j\right|_\infty,\ A^{M+1} \right) \\ &\qquad \leq c\max\left(2^{-j(M+r)},\ j^{2\alpha}2^{-2j(M-s)},\ j^\alpha 2^{-j(M+1-s)},\ 2^{-j(M+1)} \right) \\ &\qquad = c\max\left(2^{-j(M+r)},\ j^{2\alpha}2^{-2j(M-s)},\ j^\alpha 2^{-j(M+1-s)} \right). \end{aligned}$$

This shows (4.34). □

5. Convergence of Normal Multiresolution Approximation

5.1. **Introduction.** We are now ready to attack our analysis of normal multiresolution approximation. Let us first define our setting. To begin with, we are given a smooth curve Γ in $\mathbb{R}^2$. This curve can be parameterized in many ways; often we shall assume it is C^1, so that we could parameterize it by arc length. It will be more convenient for us, though, to parameterize it by one of the x- or y-coordinates. A piecewise C^1 curve can always be broken up into adjacent finite length pieces, possibly overlapping, that can be well parameterized by the x-coordinate (with, say, $|dy/dx| \le 2$) or by the y-coordinate (with, say, $|dx/dy| \le 2$); by restricting ourselves to these different pieces separately, and interchanging the names of the two coordinates, we may thus assume, without loss of generality, that the curve Γ is parameterized by its x-coordinate, so that

$$\Gamma = \{(x, \gamma(x));\ \ x \in I\},$$

where γ is a smooth function and I is an interval, a half-line or all of $\mathbb{R}$. For convenience we always assume that the definition of γ is extended to all of $\mathbb{R}$. In many cases, we shall assume that γ is at least $C^1(I)$ (corresponding to our remarks above); occasionally we shall be more general and assume only Hölder continuity with a Hölder exponent $\beta \le 1$.

Given a (possibly finite) sequence $\boldsymbol{x}_j$ in I, we define $\boldsymbol{y}_j = \gamma(\boldsymbol{x}_j)$. For every j we compute the two predictor sequences $\boldsymbol{x}^*_{j+1}$ and $\boldsymbol{y}^*_{j+1}$ using an interpolating stationary linear subdivision scheme S,

$$\boldsymbol{x}^*_{j+1} = S\boldsymbol{x}_j, \qquad \boldsymbol{y}^*_{j+1} = S\boldsymbol{y}_j.$$

Those are in general not related via the function γ, i.e. $\boldsymbol{y}^*_{j+1} \neq \gamma(\boldsymbol{x}^*_{j+1})$, but, as we will see, the sequences $\boldsymbol{y}^*_{j+1}$ and $\gamma(\boldsymbol{x}^*_{j+1})$ will be close. In a normal multiresolution we first determine, for every k, the line through the point $(x^*_{j+1,2k+1},\ y^*_{j+1,2k+1})$ that is perpendicular to the line connecting $(x_{j,k},\ y_{j,k})$ and $(x_{j,k+1},\ y_{j,k+1})$; the intersection point of this normal line and the curve Γ gives the new point $(x_{j+1,2k+1},\ y_{j+1,2k+1})$. (This is illustrated in Figure 3.) The x-coordinate of this new odd-indexed point thus satisfies

$$(x_{j+1,2k+1} - x^*_{j+1,2k+1})(\Delta\boldsymbol{x}_j)_k + (y_{j+1,2k+1} - y^*_{j+1,2k+1})(\Delta\boldsymbol{y}_j)_k = 0; \tag{5.1}$$

the even-indexed points are just taken over from the previous level, $x_{j+1,2k} = x_{j,k}$. We let the whole procedure be described by the application of the nonlinear operator N_j to the original sequence,

$$\boldsymbol{x}_{j+1} = N_j\boldsymbol{x}_j.$$

We shall always start out with a strictly increasing sequence $\boldsymbol{x}_0$, i.e. $\Delta\boldsymbol{x}_0 > 0$; in order to avoid messy difficulties with the definition of polyline approximation below we would like to have $\Delta\boldsymbol{x}_j > 0$ for all j. In general (5.1) does not always have solutions such that this is true, however. (We shall derive conditions on S, $\boldsymbol{x}_0$ and Γ to ensure this.) In any case, we shall apply the operators N_j only to sequences $\boldsymbol{x}_j$ for which $\Delta\boldsymbol{x}_j > 0$. We should also remark that (5.1) may have *several* solutions for which $\Delta\boldsymbol{x}_j > 0$. Our results below do not depend on which of these solutions is selected. For definiteness we shall assume that there is a rule established which uniquely picks out one of the solutions, should there be many. The rule could for instance be to pick the solution closest to (or furthest away from) the predicted point. When we say that the points on the next finer level are "well-defined," we mean that there exist solutions $\boldsymbol{x}_{j+1}$ with $\Delta\boldsymbol{x}_{j+1} > 0$ satisfying (5.1) and, if there are many such solutions, we implicitly assume that the rule decides which of them to select.

In order to define the convergence we wish to establish, we introduce auxiliary functions γ_j. Each γ_j interpolates linearly the values $y_{j,k}$ at the $x_{j,k}$; if $\boldsymbol{x}_j$ is strictly increasing, this is a well-defined function. Without restriction, we also assume I is the smallest interval containing all points $x_{j,k}$, so that γ_j is defined on the whole of I. The graph of γ_j, the (piecewise linear) curve Γ_j, is the normal multiresolution approximation at level j. (Note that Γ_j depends on Γ, $\boldsymbol{x}_0$ and S as well as on j.) We will then say that the normal multiresolution approximation Γ_j converges to Γ if

$$||\gamma - \gamma_j||_{L^\infty(I)} = \sup_{x\in I} |\gamma(x) - \gamma_j(x)|,$$

converges to 0 as $j \to \infty$. Now, if $\gamma \in C^\beta$ and $\tilde{\beta} = \min(\beta, 1) > 0$, then

$$\begin{aligned} &\sup_{x_{j,k} \le x \le x_{j,k+1}} |\gamma(x) - \gamma_j(x)| \\ &\qquad = \sup_{x_{j,k} \le x \le x_{j,k+1}} \left| \frac{x - x_{j,k}}{x_{j,k+1} - x_{j,k}} [\gamma(x) - \gamma(x_{j,k})] + \frac{x_{j,k+1} - x}{x_{j,k+1} - x_{j,k}} [\gamma(x_{j,k+1}) - \gamma(x)] \right| \\ &\qquad \le \sup_{|x' - x''| \le (\Delta \boldsymbol{x}_j)_k} |\gamma(x') - \gamma(x'')| \quad \le \; \Omega(\tilde{\beta}, \gamma) (\Delta \boldsymbol{x}_j)_k^{\tilde{\beta}}, \end{aligned}$$

so that

$$\|\gamma - \gamma_j\|_{L^\infty(I)} \le C \, |\Delta \boldsymbol{x}_j|_\infty^{\tilde{\beta}} .$$

The normal multiresolution approximation therefore converges to the desired limit if $\boldsymbol{x}_j$ remains strictly increasing for all j and if $|\Delta \boldsymbol{x}_j|_\infty \to 0$ as $j \to \infty$.

To prove stability and good decay estimates for the "differences", $\sqrt{(\boldsymbol{x}_j - \boldsymbol{x}_j^*)^2 + (\boldsymbol{y}_j - \boldsymbol{y}_j^*)^2}$, we will in fact need exponential convergence to 0 of $|\Delta \boldsymbol{x}_j|_\infty$ as $j \to \infty$; see below. We shall see that the rate of convergence to 0 of the differences is determined by the order $\mathcal{P}$ of S, its optimal p, μ and the smoothness of Γ.

We shall occasionally single out one particular family of interpolating subdivision schemes for the use in the prediction step: the so-called Lagrange interpolation subdivision schemes, in which the new odd-indexed points are given the values taken by a polynomial determined by several neighboring old points. For instance, in the two-point scheme, $u_{j+1,2k+1}$ is given the value at $t = 1/2$ of the *linear* polynomial that takes the values $u_{j,k}$ at $t = 0$ and $u_{j,k+1}$ at $t = 1$; in other words,

$$u_{j+1,2k+1} = \frac{1}{2}(u_{j,k} + u_{j,k+1}).$$

In the four-point scheme, $u_{j+1,2k+1}$ is given the value at $t = 1/2$ of the *cubic* that takes the values $u_{j,k-1}$, $u_{j,k}$, $u_{j,k+1}$ and $u_{j,k+2}$ at $t = -1, 0, 1, 2$ respectively, leading to

$$u_{j+1,2k+1} = \frac{9}{16}(u_{j,k} + u_{j,k+1}) - \frac{1}{16}(u_{j,k-1} + u_{j,k+2}).$$

In general, the 2ℓ-point scheme gives $u_{j+1,2k+1}$ the value at $t = 1/2$ of the $(2\ell - 1)$-degree polynomial that takes the values $u_{j,k+m}$ at $t = m$, where $m = -\ell + 1, \ldots, \ell$. We shall denote the 2ℓ-point scheme by $S_{2\ell}$. In particular the 2-point and the 4-point schemes will be denoted by S_2 and S_4:

$$\begin{aligned} (S_2 \boldsymbol{u}_j)_{2k+1} &:= \frac{1}{2}(u_{j,k} + u_{j,k+1}), \\ (S_4 \boldsymbol{u}_j)_{2k+1} &:= \frac{9}{16}(u_{j,k} + u_{j,k+1}) - \frac{1}{16}(u_{j,k-1} + u_{j,k+2}). \end{aligned}$$

Since these are all interpolating schemes, we have of course $(S_2 \boldsymbol{u}_j)_{2k} = (S_4 \boldsymbol{u}_j)_{2k} = (S_{2\ell} \boldsymbol{u}_j)_{2k} = u_{j,k}$.

When the prediction step is computed by means of S_2, i.e.

$$\boldsymbol{x}_{j+1}^* = S_2 \boldsymbol{x}_j, \qquad \boldsymbol{y}_{j+1}^* = S_2 \boldsymbol{y}_j,$$

it turns out that the analysis of normal multiresolution approximation is especially simple. We shall see below that we always have convergence if the function γ is in $C^\beta(\mathbb{R})$, with $\beta > 0$, without any restrictions on the initial data other than $\Delta \boldsymbol{x}_0 > 0$. For other prediction subdivision schemes S, however, even for $S = S_{2\ell}$ with $\ell > 1$, convergence is not as automatic; in general we need to impose restrictions on both $\boldsymbol{x}_0$ and on S. The special property that simplifies the analysis for $S = S_2$ is the monotonicity of S_2: if $\Delta \boldsymbol{u} > 0$, then $\Delta S_2 \boldsymbol{u} > 0$. In general linear subdivision schemes do not map arbitrary increasing sequences to increasing sequences. For instance, S_4 is not monotone: if $u_\ell = \ell$ for $\ell \le 1$, $u_\ell = 11 + \ell$ for $\ell \ge 2$, then $(S_4 \boldsymbol{u})_1 = -1/16 < (S_4 \boldsymbol{u})_0 = u_0 = 0$. In this example, there is a sudden large jump in the ratio $(\Delta \boldsymbol{u})_{k+1}/(\Delta \boldsymbol{u})_k$ as k crosses from 0 to 1, which causes $S_4 \boldsymbol{u}$ to be no longer increasing. We shall keep track of such non-uniformity by means of the *non-uniformity measure* $\mathcal{N}$ [15]. The general topic of monotonicity preserving interpolating subdivision schemes is studied in [15], where several non-linear monotonicity preserving schemes are introduced. For a given sequence $\boldsymbol{x}$ we define the function $\mathcal{N}$ as

$$\mathcal{N}(\boldsymbol{x}) := \sup_k \max\left(\frac{|(\Delta \boldsymbol{x})_k|}{|(\Delta \boldsymbol{x})_{k+1}|}, \frac{|(\Delta \boldsymbol{x})_{k+1}|}{|(\Delta \boldsymbol{x})_k|}\right).$$

It turns out [19] that if one restricts oneself to strictly increasing sequences with $\mathcal{N}(\boldsymbol{x}) \leq 3 + 2\sqrt{2}$, then $S_4\boldsymbol{x}$ is strictly increasing again; moreover, $\mathcal{N}(S_4\boldsymbol{x}) \leq 3 + 2\sqrt{2}$, so that all $S_4^j\boldsymbol{u}$ will be strictly increasing. This leads us to define the notion of weak monotonicity:

Definition 5.1. *We call a subdivision scheme* weakly monotone *with bound R if for every strictly increasing $\boldsymbol{x}$ with*

$$\mathcal{N}(\boldsymbol{x}) \leq R,$$

$S\boldsymbol{x}$ is strictly increasing as well, and satisfies

$$\mathcal{N}(S\boldsymbol{x}) \leq R. \tag{5.2}$$

In fact, S_4 has the stronger property that if $\mathcal{N}(\boldsymbol{x}) \leq 3+2\sqrt{2}$, then $\mathcal{N}(S_4\boldsymbol{x}) \leq \mathcal{N}(\boldsymbol{x})$ [19]. For completeness we include a proof of this as outlined to us by Ruud van Damme in Appendix B. We shall give this stronger property a special name as well.

Definition 5.2. *We call a subdivision scheme* weakly contractive *with bound R if for every strictly increasing $\boldsymbol{x}$ with*

$$\mathcal{N}(\boldsymbol{x}) \leq R,$$

$S\boldsymbol{x}$ is strictly increasing as well, and satisfies

$$\mathcal{N}(S\boldsymbol{x}) \leq \mathcal{N}(\boldsymbol{x}). \tag{5.3}$$

In our proofs, we will require this stronger notion of weak contractivity. In fact, S_6 and S_8 are also weakly contractive, see [17].

The main goal of this section is to show that if S is weakly contractive, if both $|\Delta\boldsymbol{x}_0|_\infty$ and $\mathcal{N}(\boldsymbol{x}_0)$ are sufficiently small, and if $\gamma \in C^\beta(\mathbb{R})$ with $\beta > 1$, then $|\Delta\boldsymbol{x}_j|_\infty$ converges to zero exponentially. Before proving this main result, Theorem 5.7 in Section 5.3 below, we prove several technical lemmas, bundled together in Section 5.2.

We assume S is a linear, bounded, local and interpolating subdivision scheme. As before, $\mathcal{P} \geq 1$ denotes the order of S; we pick $p \in \{1, \ldots, \mathcal{P}\}$ optimal for S, and $\mu \geq 0$ to be p-suitable. (See Section 4.1.) We denote by B the width of S, as given in Section 2. Throughout this section β will be the Hölder exponent of the function γ, i.e. $\gamma \in C^\beta(\mathbb{R})$.

5.2. **Preliminary Lemmas.** These lemmas will only concern one refinement step in the normal multiresolution scheme. We denote the initial level by $\boldsymbol{x}$ and the next level by $\tilde{\boldsymbol{x}}$; with this notation (5.1) becomes

$$(\tilde{x}_{2k+1} - x^*_{2k+1})(\Delta\boldsymbol{x})_k + (\gamma(\tilde{x}_{2k+1}) - y^*_{2k+1})(\Delta\boldsymbol{y})_k = 0. \tag{5.4}$$

Mostly we leave out the index k when it is understood anyway; we shall also use the shorthand notation

$$\Delta x = (\Delta\boldsymbol{x})_k, \quad \Delta y = (\Delta\boldsymbol{y})_k, \quad x^* = (S\boldsymbol{x})_{2k+1}, \quad y^* = (S\boldsymbol{y})_{2k+1} = (S\gamma(\boldsymbol{x}))_{2k+1}, \quad \tilde{x} = \tilde{x}_{2k+1}. \tag{5.5}$$

We also define the help sequences $\boldsymbol{\varepsilon}^x$, $\boldsymbol{\varepsilon}^y$ and $\boldsymbol{\alpha}$ as

$$\varepsilon^x_k = ((S - S_2)\boldsymbol{x})_{2k+1}, \qquad \varepsilon^y_k = ((S - S_2)\boldsymbol{y})_{2k+1}, \qquad \boldsymbol{\alpha} = 2\frac{\boldsymbol{\varepsilon}^x\Delta\boldsymbol{x} + \boldsymbol{\varepsilon}^y\Delta\boldsymbol{y}}{(\Delta\boldsymbol{x})^2 + (\Delta\boldsymbol{y})^2}. \tag{5.6}$$

The divergence of the predictor S from the two-point scheme S_2 is measured by $\boldsymbol{\varepsilon}^x$ and $\boldsymbol{\varepsilon}^y$, while the magnitude of $\boldsymbol{\alpha}$ determines whether the normal scheme will pierce the curve in between points from the preceding level, hence whether monotonocity of $\boldsymbol{x}$ will be preserved by $\tilde{\boldsymbol{x}}$. (See Figure 3.) For simplicity, we set

$$\varepsilon^x = \varepsilon^x_k = x^* - \frac{x_k + x_{k+1}}{2}, \qquad \varepsilon^y = \varepsilon^y_k = y^* - \frac{y_k + y_{k+1}}{2}, \qquad \alpha = \alpha_k. \tag{5.7}$$

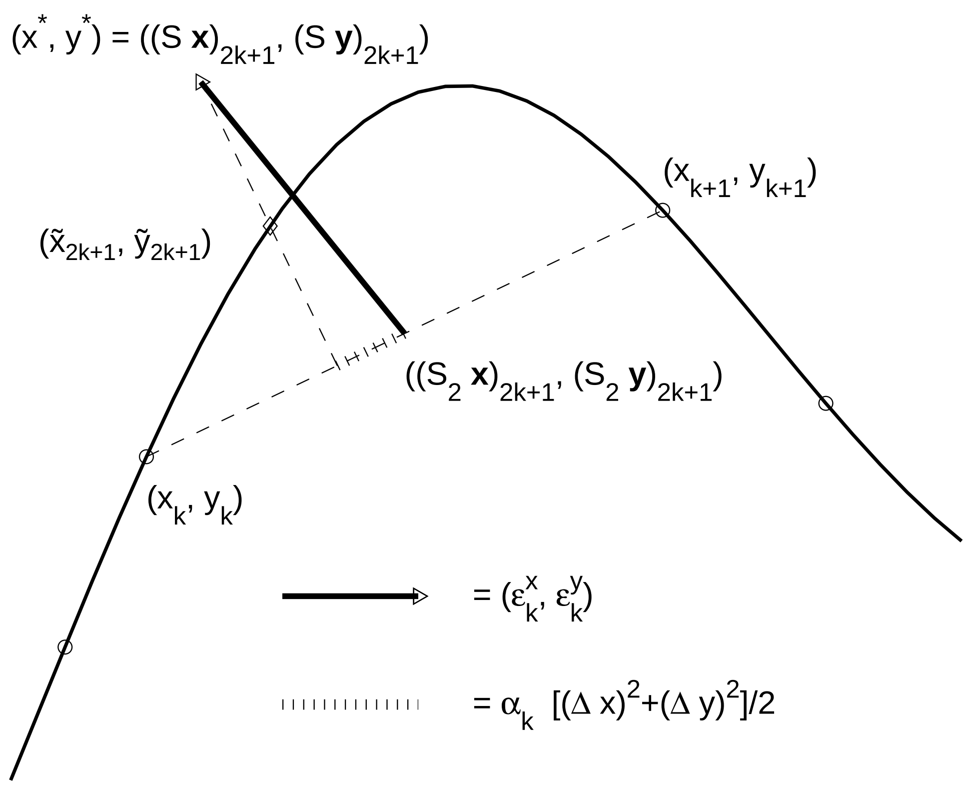

FIGURE 3. The help sequences $\boldsymbol{\varepsilon}^x$, $\boldsymbol{\varepsilon}^y$ and $\boldsymbol{\alpha}$ introduced in Section 5.2.

Lemma 5.3. *Suppose $\boldsymbol{x}$ is a strictly increasing sequence and $\beta > 0$. If*

$$|\boldsymbol{\alpha}|_\infty \leq b < 1, \tag{5.8}$$

then $\tilde{\boldsymbol{x}}$ is well-defined and strictly increasing. If $\beta \geq 1$ there is a constant δ independent of k,

$$\delta = 1 - \left(\frac{1-b}{2 + c'_\gamma \, |\Delta \boldsymbol{x}|_\infty^r}\right), \qquad c'_\gamma = \Omega(r, \gamma'), \qquad r = \min(\beta - 1, 1), \qquad \frac{1}{2} \leq \delta < 1, \tag{5.9}$$

such that

$$\max((\Delta \tilde{\boldsymbol{x}})_{2k},\ (\Delta \tilde{\boldsymbol{x}})_{2k+1}) \leq \delta (\Delta \boldsymbol{x})_k, \qquad \forall k. \tag{5.10}$$

If $0 < \beta \leq 1^-$, we obtain

$$\max((\Delta \tilde{\boldsymbol{x}})_{2k},\ (\Delta \tilde{\boldsymbol{x}})_{2k+1}) \leq (\Delta \boldsymbol{x})_k \left(1 - \left(\frac{1-b}{2 + c_\gamma (\Delta \boldsymbol{x})_k^{\beta-1}}\right)^{1/\beta}\right), \qquad \forall k, \tag{5.11}$$

where $c_\gamma = \Omega(\beta, \gamma)$.

Proof. We fix the index k and use the shorthand notation of (5.5, 5.7). Then we introduce the function

$$f(t) = \frac{(t\Delta x + x_k - x^*)\,\Delta x + (\gamma\,(t\Delta x + x_k) - y^*)\Delta y}{(\Delta x)^2 + (\Delta y)^2} \in C^\beta[0,1],$$

which is well-defined since $\boldsymbol{x}$ is well-defined and strictly increasing. The equation (5.4) for $\tilde{x}$,

$$(\tilde{x}-x^*)\Delta x+(\gamma(\tilde{x})-y^*)\Delta y=0,$$

can then be recast as

$$f\left(\frac{\tilde{x}-x_k}{\Delta x}\right)=0. \tag{5.12}$$

Moreover,

$$f(0)=-\frac{1+\alpha}{2}\leq-\frac{1-b}{2}<0,\qquad f(1)=\frac{1-\alpha}{2}\geq\frac{1-b}{2}>0;$$

by continuity there is a root to f in the interval $(0,1)$, so we can indeed take $\tilde{x}\in(x_k,x_{k+1})$. This shows that $\tilde{\boldsymbol{x}}$ is strictly increasing. Set $s=\min(\beta,1)$. By (5.12),

$$|f(0)|\leq c_f\left(\frac{\tilde{x}-x_k}{\Delta x}\right)^s,\qquad |f(1)|\leq c_f\left(\frac{x_{k+1}-\tilde{x}}{\Delta x}\right)^s,\qquad c_f=\max(1,\Omega(s,f)). \tag{5.13}$$

Letting δ satisfy

$$1-\left(\frac{1-b}{2c_f}\right)^{1/s}\leq\delta<1, \tag{5.14}$$

we have

$$\begin{aligned}1-\delta &\leq \left(\frac{1+\alpha}{2c_f}\right)^{1/s}=\left|\frac{f(0)}{c_f}\right|^{1/s}\leq\frac{\tilde{x}-x_k}{\Delta x}=1-\frac{x_{k+1}-\tilde{x}}{\Delta x}\\ &\leq 1-\left|\frac{f(1)}{c_f}\right|^{1/s}=1-\left(\frac{1-\alpha}{2c_f}\right)^{1/s}\leq\delta,\end{aligned} \tag{5.15}$$

from which it follows that

$$\max((\Delta\tilde{\boldsymbol{x}})_{2k},\ (\Delta\tilde{\boldsymbol{x}})_{2k+1})=\Delta x\max\left(\frac{\tilde{x}-x_k}{\Delta x},\frac{x_{k+1}-\tilde{x}}{\Delta x}\right)\leq\delta\Delta x. \tag{5.16}$$

Assume now that $\beta\geq1$. Then $s=1$ and $c_f=\max(1,|f'|_\infty)$. Since $\gamma\in C^1(\mathbb{R})$ there is a $\xi\in[x_k,x_{k+1}]$ such that $\gamma'(\xi)=\Delta y/\Delta x$, and therefore,

$$f'(t)=\frac{(\Delta x)^2+\gamma'(t\Delta x+x_k)\Delta y\Delta x}{(\Delta x)^2+(\Delta y)^2}=1+\frac{(\gamma'(t\Delta x+x_k)-\gamma'(\xi))\Delta y\Delta x}{(\Delta x)^2+(\Delta y)^2}.$$

Consequently, for $0\leq t\leq1$ and $c'_\gamma=\Omega(r,\gamma')$,

$$|f'(t)|\leq1+c'_\gamma|t\Delta x+x_k-\xi|^r\frac{|\Delta y|\Delta x}{(\Delta x)^2+(\Delta y)^2}\leq1+(\Delta x)^r\frac{c'_\gamma}{2}\leq1+\frac{1}{2}c'_\gamma\,|\Delta\boldsymbol{x}|_\infty^r=c_f.$$

This estimate together with (5.14) gives (5.9). Suppose now that $0<\beta\leq1-$ so that $s=\beta$. Then

$$\begin{aligned}\Omega(s,f) &= \sup_{0\leq t_1<t_0\leq1}\frac{\left|(t_0-t_1)(\Delta x)^2+(\gamma(t_0\Delta x+x_k)-\gamma(t_1\Delta x+x_k))\Delta y\right|}{[(\Delta x)^2+(\Delta y)^2]\ |t_0-t_1|^\beta}\\ &\leq \frac{|t_1-t_0|^{1-\beta}(\Delta x)^2+c_\gamma(\Delta x)^\beta|\Delta y|}{(\Delta x)^2+(\Delta y)^2}\leq\frac{(\Delta x)^2}{(\Delta x)^2+(\Delta y)^2}+\frac{c_\gamma}{2}(\Delta x)^{\beta-1}\frac{2\Delta x|\Delta y|}{(\Delta x)^2+(\Delta y)^2}\\ &\leq 1+\frac{c_\gamma}{2}(\Delta x)^{\beta-1}=c_f.\end{aligned}$$

The estimate together with (5.13, 5.14, 5.16) gives (5.11). □

Next, we show that upper bounds on $\boldsymbol{\alpha}$, needed to apply Lemma 5.3, can be derived from the data:

Lemma 5.4. *Let $\boldsymbol{x}$ be a strictly increasing sequence. Suppose that $\beta > 1$ and that there is a constant λ such that*

$$\left|\frac{\boldsymbol{\varepsilon}^x}{\Delta \boldsymbol{x}}\right|_\infty \leq \lambda \leq 1. \tag{5.17}$$

Then there is a constant c that only depends on γ and on S, such that

$$|\boldsymbol{\alpha}|_\infty \leq 2\lambda + c\,\mathcal{N}(\boldsymbol{x})^B\,|\Delta \boldsymbol{x}|_\infty^r, \qquad r = \min(\beta - 1, 1). \tag{5.18}$$

Proof. We fix k and as in Lemma 5.3 we use the shorthands (5.5, 5.7). Then

$$\begin{aligned} |\alpha| &= 2\frac{|\varepsilon^x \Delta x + \varepsilon^y \Delta y|}{(\Delta x)^2 + (\Delta y)^2} = 2\frac{\left|\varepsilon^x \Delta x + \varepsilon^x \frac{(\Delta y)^2}{\Delta x} + \Delta y \left(\varepsilon^y - \varepsilon^x \frac{\Delta y}{\Delta x}\right)\right|}{(\Delta x)^2 + (\Delta y)^2} \\ &\leq 2\lambda + \frac{2|\Delta y| \left|\varepsilon^y - \varepsilon^x \frac{\Delta y}{\Delta x}\right|}{(\Delta x)^2 + (\Delta y)^2}. \end{aligned} \tag{5.19}$$

Moreover,

$$\begin{aligned} \varepsilon^y - \varepsilon^x \frac{\Delta y}{\Delta x} &= (S\gamma(\boldsymbol{x}) - S_2\gamma(\boldsymbol{x}))_{2k+1} - \varepsilon^x \frac{\Delta y}{\Delta x} \\ &= (S\gamma(\boldsymbol{x}) - \gamma(S\boldsymbol{x}))_{2k+1} + (\gamma(S_2\boldsymbol{x}) - S_2\gamma(\boldsymbol{x}))_{2k+1} \\ &\quad + (\gamma(S\boldsymbol{x}) - \gamma(S_2\boldsymbol{x}))_{2k+1} - \varepsilon^x \frac{\Delta y}{\Delta x}. \end{aligned}$$

By Lemma 4.6, with $M = 1$, the first two terms can be bounded by

$$c \max_{\ell \in I_{2k+1}} |(\Delta \boldsymbol{x})_\ell|^{1+r} \leq c\,\mathcal{N}(\boldsymbol{x})^B \Delta x\,|\Delta \boldsymbol{x}|_\infty^r.$$

For the last terms we resort to Taylor expansion around $(S_2\boldsymbol{x})_{2k+1}$. Since $\gamma \in C^{1+r}$, there is a $\xi \in [x_k, x_{k+1}]$ and R such that

$$\begin{aligned} (\gamma(S\boldsymbol{x}) - \gamma(S_2\boldsymbol{x}))_{2k+1} - \varepsilon^x \frac{\Delta y}{\Delta x} &= (S\boldsymbol{x} - S_2\boldsymbol{x})_{2k+1}\gamma'(S_2\boldsymbol{x})_{2k+1} + R(\boldsymbol{x})_{2k+1} - \varepsilon^x \gamma'(\xi) \\ &= \varepsilon^x(\gamma'(x^*) - \gamma'(\xi)) + R(\boldsymbol{x})_{2k+1}, \end{aligned}$$

where

$$|(R(\boldsymbol{x}))_{2k+1}| \leq C|(S\boldsymbol{x} - S_2\boldsymbol{x})_{2k+1}|^{1+r} = C|\varepsilon^x|^{1+r}.$$

Entering these estimates into (5.19) gives

$$\begin{aligned} |\alpha| &\leq 2\lambda + \frac{2|\Delta y| \left[c\,\mathcal{N}(\boldsymbol{x})^B \Delta x\,|\Delta \boldsymbol{x}|_\infty^r + |\varepsilon^x||\gamma'(x^*) - \gamma'(\xi)| + C|\varepsilon^x|^{1+r}\right]}{(\Delta x)^2 + (\Delta y)^2} \\ &\leq 2\lambda + \left[c\,\mathcal{N}(\boldsymbol{x})^B\,|\Delta \boldsymbol{x}|_\infty^r + \lambda\Omega(r, \gamma')|x^* - \xi|^r + C\lambda^{1+r}\,|\Delta \boldsymbol{x}|_\infty^r\right] \frac{2|\Delta y|\Delta x}{(\Delta x)^2 + (\Delta y)^2} \\ &\leq 2\lambda + c\,\mathcal{N}(\boldsymbol{x})^B\,|\Delta \boldsymbol{x}|_\infty^r. \end{aligned}$$

□

It easily follows from Lemmas 5.3 and 5.4 that if $\left|\frac{\boldsymbol{\varepsilon}^x}{\Delta \boldsymbol{x}}\right|_\infty \leq \lambda < 1/2$, and if $\mathcal{N}(\boldsymbol{x})^B\,|\Delta \boldsymbol{x}|_\infty^r$ is sufficiently small, then $\tilde{\boldsymbol{x}}$ is well-defined and strictly increasing. We shall assume the existence and monotonicity $\tilde{\boldsymbol{x}}$ in the two lemmas that follow. Note that $\mathcal{N}(\boldsymbol{x})$ plays a role in the bound (5.18) on $\boldsymbol{\alpha}$. In order to iterate these estimates over more than one level, we will therefore need to bound $\mathcal{N}(\tilde{\boldsymbol{x}})$ as well. We start by an estimate on $|\tilde{\boldsymbol{x}} - S\boldsymbol{x}|_\infty$:

Lemma 5.5. *Let S, $\mathcal{P}$, γ and β be as prescribed at the end of Section 5.1. If $\boldsymbol{x}$ and $\tilde{\boldsymbol{x}}$ are both strictly increasing and if*

$$\beta \geq 1 \text{ and } \gamma' \text{ is uniformly continuous,} \tag{5.20}$$

then for sufficiently small $|\Delta \boldsymbol{x}|_\infty$,

$$|(\tilde{\boldsymbol{x}} - S\boldsymbol{x})_{2k+1}| \le |(S\gamma(\boldsymbol{x}) - \gamma(S\boldsymbol{x}))_{2k+1}|, \qquad \forall k,$$

and

$$|\tilde{\boldsymbol{x}} - S\boldsymbol{x}|_\infty \le |S\gamma(\boldsymbol{x}) - \gamma(S\boldsymbol{x})|_\infty . \tag{5.21}$$

In particular, if $\beta > 1$ *and* $r = \min(\beta - 1, 1)$*, then* (5.21) *holds if*

$$|\Delta \boldsymbol{x}|_\infty \le \Big[2\, |\gamma'|_\infty\, \Omega(r, \gamma')\Big]^{-1/r} C^{-1}, \tag{5.22}$$

where C *is the constant of Proposition* 3.1.

Proof. Since S is interpolating, the second estimate (5.21) follows trivially from the first. We fix k and as before introduce the shorthand (5.5). Since $\gamma \in C^1$, there is a $\xi_1 \in [\min(\tilde{x}, x^*), \max(\tilde{x}, x^*)]$ such that

$$\gamma'(\xi_1) = \frac{\gamma(\tilde{x}) - \gamma(x^*)}{\tilde{x} - x^*}.$$

Moreover, by the construction of the normal scheme we can find ξ_2 satisfying

$$\gamma'(\xi_2) = \frac{\gamma(x_{k+1}) - \gamma(x_k)}{x_{k+1} - x_k} = \frac{\tilde{x} - x^*}{y^* - \gamma(\tilde{x})}$$

and $\xi_2 \in [x_k, x_{k+1}]$ since $\boldsymbol{x}$ is increasing. Then,

$$\begin{aligned}(\tilde{x} - x^*)^2 &= (\gamma(\tilde{x}) - \gamma(x^*))(y^* - \gamma(\tilde{x})) + (\tilde{x} - x^*)(y^* - \gamma(\tilde{x}))(\gamma'(\xi_2) - \gamma'(\xi_1)) \\ &= (\gamma(\tilde{x}) - \gamma(x^*))(y^* - \gamma(\tilde{x})) + (y^* - \gamma(\tilde{x}))^2 \gamma'(\xi_2)(\gamma'(\xi_2) - \gamma'(\xi_1)).\end{aligned}$$

We furthermore note that $\tilde{\boldsymbol{x}}$ is increasing, $x_k \le \tilde{x} \le x_{k+1}$, and that $\mathcal{P} \ge 1$, so

$$\begin{aligned}|\xi_1 - \xi_2| &\le \max(|\tilde{x} - x_k|, |\tilde{x} - x_{k+1}|, |x^* - x_k|, |x^* - x_{k+1}|) \\ &\le \max(|\Delta \boldsymbol{x}|_\infty, \max_{\ell \in I_{2k+1}} |x_\ell - (S\boldsymbol{x})_{2k+1}|) \le C\, |\Delta \boldsymbol{x}|_\infty ,\end{aligned}$$

by (3.2) in Proposition 3.1. Then, since $\gamma'(x)$ is uniformly continuous,

$$|\gamma'(\xi_2)(\gamma'(\xi_2) - \gamma'(\xi_1))| \le |\gamma'|_\infty |\gamma'(\xi_2) - \gamma'(\xi_1)| \le 1/2,$$

for $|\Delta \boldsymbol{x}|_\infty$ small enough. In particular, if $\beta > 1$, and (5.22) is satisfied, then

$$|\gamma'|_\infty |\gamma'(\xi_2) - \gamma'(\xi_1)| \le \Omega(r, \gamma')|\xi_2 - \xi_1|^r\, |\gamma'|_\infty \le \Omega(r, \gamma') C^r\, |\Delta \boldsymbol{x}|_\infty^r\, |\gamma'|_\infty \le 1/2.$$

Hence,

$$\begin{aligned}(\tilde{x} - x^*)^2 &\le (\gamma(\tilde{x}) - \gamma(x^*))(y^* - \gamma(\tilde{x})) + \frac{(y^* - \gamma(\tilde{x}))^2}{2} \\ &\le (\gamma(\tilde{x}) - \gamma(x^*))(y^* - \gamma(\tilde{x})) + \frac{(y^* - \gamma(\tilde{x}))^2}{2} + \frac{(\gamma(\tilde{x}) - \gamma(x^*))^2}{2} \\ &= (y^* - \gamma(x^*))^2.\end{aligned}$$

This proves the lemma. □

We now use this to derive a bound on $\mathcal{N}(\tilde{\boldsymbol{x}})$:

Lemma 5.6. *Let* $\boldsymbol{x}$, $S\boldsymbol{x}$ *and* $\tilde{\boldsymbol{x}}$ *all be strictly increasing sequences and suppose* $\beta > 1$*. Then, if* (5.22) *is satisfied with* $r = \min(\beta - 1, 1)$*, and if*

$$C\, \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\, |\Delta \boldsymbol{x}|_\infty^r \le 1/4, \tag{5.23}$$

where C *is the constant in* (4.20) *in Lemma* 4.6*, then*

$$\mathcal{N}(\tilde{\boldsymbol{x}}) \le \mathcal{N}(S\boldsymbol{x})(1 + 8C\, \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\, |\Delta \boldsymbol{x}|_\infty^r). \tag{5.24}$$

Proof. Since S and the normal scheme are both interpolating, $\tilde{x}_{2k} = (S\boldsymbol{x})_{2k}$, and, with $\ell = 0, 1$,

$$|(\Delta\tilde{\boldsymbol{x}} - \Delta S\boldsymbol{x})_{2k+\ell}| = |\tilde{x}_{2k+1} - (S\boldsymbol{x})_{2k+1}|.$$

By our assumptions we can use Lemma 5.5 and then Lemma 4.6 with $M = 1$ to get

$$|(\Delta\tilde{\boldsymbol{x}} - \Delta S\boldsymbol{x})_{2k+\ell}| \leq |(S\gamma(\boldsymbol{x}) - \gamma(S\boldsymbol{x}))_{2k+1}| \leq C \max_{\ell \in I_{2k+1}} |(\Delta\boldsymbol{x})_\ell|^{1+r} \leq C\ \mathcal{N}(\boldsymbol{x})^B (\Delta\boldsymbol{x})_k\ |\Delta\boldsymbol{x}|_\infty^r\,. \tag{5.25}$$

Moreover, since $\Delta S\boldsymbol{x} > 0$,

$$(\Delta S\boldsymbol{x})_{2k+\ell} \geq \frac{(\Delta S\boldsymbol{x})_{2k} + (\Delta S\boldsymbol{x})_{2k+1}}{2\mathcal{N}(S\boldsymbol{x})} = \frac{(\Delta\boldsymbol{x})_k}{2\mathcal{N}(S\boldsymbol{x})}. \tag{5.26}$$

Using (5.25, 5.26) we then get

$$\frac{|(\Delta\tilde{\boldsymbol{x}} - \Delta S\boldsymbol{x})_{2k+\ell}|}{(\Delta S\boldsymbol{x})_{2k+\ell}} \leq 2C\ \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\ |\Delta\boldsymbol{x}|_\infty^r\,.$$

Since this quantity is smaller than $1/2$ we have, now with $\ell = \pm 1$,

$$\begin{aligned}
\frac{(\Delta\tilde{\boldsymbol{x}})_k}{(\Delta\tilde{\boldsymbol{x}})_{k+\ell}} &= \frac{(\Delta S\boldsymbol{x})_k + (\Delta\tilde{\boldsymbol{x}} - \Delta S\boldsymbol{x})_k}{(\Delta S\boldsymbol{x})_{k+\ell} + (\Delta\tilde{\boldsymbol{x}} - \Delta S\boldsymbol{x})_{k+\ell}} \\
&\leq \frac{(\Delta S\boldsymbol{x})_k}{(\Delta S\boldsymbol{x})_{k+\ell}} \left(\frac{1 + 2C\ \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\ |\Delta\boldsymbol{x}|_\infty^r}{1 - 2C\ \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\ |\Delta\boldsymbol{x}|_\infty^r}\right) \\
&\leq \frac{(\Delta S\boldsymbol{x})_k}{(\Delta S\boldsymbol{x})_{k+\ell}} \left(1 + 8C\ \mathcal{N}(S\boldsymbol{x})\mathcal{N}(\boldsymbol{x})^B\ |\Delta\boldsymbol{x}|_\infty^r\right),
\end{aligned}$$

where we have used $(1 + 2t)/(1 - 2t) \leq 1 + 8t$ for $0 \leq t \leq 1/4$. This proves (5.24). □

Remark: One could also replace $\mathcal{N}(\boldsymbol{x})^B$ in these last two lemmas by the possibly smaller quantity,

$$\mathcal{N}_B(\boldsymbol{x}) := \sup_k \left\{\max\left(\frac{|(\Delta\boldsymbol{x})_k|}{|(\Delta\boldsymbol{x})_{k+\ell}|}, \frac{|(\Delta\boldsymbol{x})_{k+\ell}|}{|(\Delta\boldsymbol{x})_k|}\right);\ \ell = 1, \ldots, B\right\}.$$

Nowhere in this subsection have we assumed weak monotonicity or weak contractivity for S. In the next subsection, we shall introduce this assumption to set up iterative estimation. In summary, we will have the following argument:

- Start with $\boldsymbol{x}$ strictly increasing, with $\mathcal{N}(\boldsymbol{x}) < R$.
- Assume S is weakly contractive with bound R, so that $S\boldsymbol{x}$ is also strictly increasing, with $\mathcal{N}(S\boldsymbol{x}) \leq \mathcal{N}(\boldsymbol{x}) < R$.
- If $\left|\frac{\boldsymbol{\varepsilon}^{\boldsymbol{x}}}{\Delta\boldsymbol{x}}\right|_\infty \leq \lambda \leq 1/2$ and if $\mathcal{N}(\boldsymbol{x})^{B+1}\ |\Delta\boldsymbol{x}|_\infty^r$ is sufficiently small, we have $|\boldsymbol{\alpha}|_\infty \leq b < 1$ by Lemma 5.4, so that, by Lemma 5.3, $\tilde{\boldsymbol{x}}$ is well-defined and strictly increasing.
- By Lemma 5.6, we have a bound on $\mathcal{N}(\tilde{\boldsymbol{x}})$ which will allow us to start on the next step, as we shall see in more detail in the next subsection.

5.3. **Convergence Theorems.** In this section, we return to the iterated normal multiresolution approximation: we consider successively refined $\boldsymbol{x}_j$ again, and the corresponding $\boldsymbol{y}_j$, $\boldsymbol{x}_j^*$ and $\boldsymbol{y}_j^*$ as defined in Section 5.1. For every level j, we use again the notation $\boldsymbol{x}_{j+1}$ for the result of the normal refinement of $\boldsymbol{x}_j$, corresponding to the refinement $\boldsymbol{x} \to \tilde{\boldsymbol{x}}$ in the preceding subsection. We start by proving exponential convergence of $|\Delta\boldsymbol{x}_j|_\infty$, thereby establishing convergence of the normal multiresolution approximation.

Theorem 5.7. *Let S be a weakly contractive, linear, bounded, local and interpolating, subdivision scheme with bound R and of order $\mathcal{P} \geq 1$. Suppose that $\gamma \in C^\beta(\mathbb{R})$ with $\beta > 1$, that $\boldsymbol{x}_0$ is strictly increasing, and that*

$$\mathcal{N}(\boldsymbol{x}_0) < R. \tag{5.27}$$

Put $r = \min(\beta - 1, 1)$ *and pick* δ *so that*

$$\frac{R}{R+1} < \delta < 1. \tag{5.28}$$

If

$$R^{B+1} \left|\Delta \boldsymbol{x}_0\right|_\infty^r \textit{ is sufficiently small}, \tag{5.29}$$

then $\boldsymbol{x}_j$ *is strictly increasing for all* $j \geq 0$,

$$\mathcal{N}(\boldsymbol{x}_j) \leq R, \qquad \forall j \geq 0, \tag{5.30}$$

and

$$\left|\Delta \boldsymbol{x}_j\right|_\infty \leq \delta^j \left|\Delta \boldsymbol{x}_0\right|_\infty, \qquad \forall j \geq 0. \tag{5.31}$$

Proof. For simplicity of notation we set $\nu_j = \mathcal{N}(\boldsymbol{x}_j)$. We also let η and b be real numbers satisfying

$$\delta^r < \eta < 1 \quad \text{ and } \quad \frac{R-1}{R+1} < b < 2\delta - 1, \tag{5.32}$$

and assume that $R^{B+1} \left|\Delta \boldsymbol{x}_0\right|_\infty^r \leq \epsilon$, with $\epsilon > 0$ to be determined in the proof. The proof works by induction. At every induction step we shall prove that

$$\boldsymbol{x}_j \text{ is strictly increasing}, \tag{5.33}$$

$$\nu_j \leq R, \tag{5.34}$$

$$\left|\Delta \boldsymbol{x}_j\right|_\infty \leq \delta^j \left|\Delta \boldsymbol{x}_0\right|_\infty \tag{5.35}$$

$$\nu_j^{B+1} \left|\Delta \boldsymbol{x}_j\right|_\infty^r \leq \epsilon \eta^j. \tag{5.36}$$

For $j = 0$, (5.33, 5.34, 5.35, 5.36) are all satisfied. We shall now prove that if (5.33, 5.34, 5.35, 5.36), are satisfied for $j = 0, \ldots, n-1$, then they must hold also for $j = n$.

First of all, since $\boldsymbol{x}_{n-1}$ is strictly increasing, and $\nu_{n-1} \leq R$, then $S\boldsymbol{x}_{n-1}$ is strictly increasing as well and $\mathcal{N}(S\boldsymbol{x}_{n-1}) \leq \nu_{n-1} \leq R$. In order to invoke Lemma 5.3 to derive monotonicity for $\boldsymbol{x}_n$, we first need an appropriate bound on $\boldsymbol{\alpha}_{n-1}$, the $\boldsymbol{\alpha}$-sequence corresponding to $\boldsymbol{x}_{n-1}$. To obtain this bound, we shall use Lemma 5.4. Set $\varepsilon^x_{n-1,k} = \varepsilon_k^{x_{n-1}}$, with the right hand side defined by (5.6). Since S is interpolating, $(\Delta \boldsymbol{x}_{n-1})_k = (\Delta S\boldsymbol{x}_{n-1})_{2k} + (\Delta S\boldsymbol{x}_{n-1})_{2k+1}$. Set, just for the next few lines,

$$m_k = \min\Big[\Delta S\boldsymbol{x}_{n-1})_{2k},\ (\Delta S\boldsymbol{x}_{n-1})_{2k+1}\Big], \qquad M_k = \max\Big[\Delta S\boldsymbol{x}_{n-1})_{2k},\ (\Delta S\boldsymbol{x}_{n-1})_{2k+1}\Big].$$

Then,

$$\begin{aligned}
\sup_k \frac{|\varepsilon^x_{n-1,k}|}{(\Delta \boldsymbol{x}_{n-1})_k} &= \sup_k \frac{1}{2}\frac{|(\Delta S\boldsymbol{x}_{n-1})_{2k+1} - (\Delta S\boldsymbol{x}_{n-1})_{2k}|}{(\Delta S\boldsymbol{x}_{n-1})_{2k+1} + (\Delta S\boldsymbol{x}_{n-1})_{2k}} \\
&\leq \sup_k \frac{1}{2}\frac{M_k/m_k - 1}{M_k/m_k + 1} = \frac{1}{2}\frac{\nu_{n-1} - 1}{\nu_{n-1} + 1} \leq \frac{1}{2}\frac{R-1}{R+1},
\end{aligned}$$

where we have used in the last step that $(x-1)/(x+1)$ is increasing on $[1, \infty)$. It then follows from Lemma 5.4 that

$$\left|\boldsymbol{\alpha}_{n-1}\right|_\infty \leq \frac{R-1}{R+1} + C_1\, \nu_{n-1}^B \left|\Delta \boldsymbol{x}_{n-1}\right|_\infty^r,$$

where the constant C_1 depends only on γ and S. Since, by the induction hypothesis (5.36), and the fact that $\mathcal{N}(\boldsymbol{u}) \geq 1$ for all sequences $\boldsymbol{u}$,

$$\nu_{n-1}^B \left|\Delta \boldsymbol{x}_{n-1}\right|_\infty^r \leq \nu_{n-1}^{B+1} \left|\Delta \boldsymbol{x}_{n-1}\right|_\infty^r \leq \epsilon \eta^{n-1} \leq \epsilon, \tag{5.37}$$

we therefore have

$$\left|\boldsymbol{\alpha}_{n-1}\right|_\infty \leq \frac{R-1}{R+1} + C_1 \epsilon.$$

If

$$\epsilon \le C_1^{-1}\left(b - \frac{R-1}{R+1}\right), \tag{5.38}$$

which we shall assume henceforth, we have thus $|\boldsymbol{\alpha}_{n-1}|_\infty \le b < 1$. Note, the right hand side of (5.38) is positive by (5.32). By Lemma 5.3, we conclude that $\boldsymbol{x}_n$ is well-defined and strictly increasing, establishing (5.33) for $j = n$. We now proceed to prove (5.34). By (5.37),

$$|\Delta \boldsymbol{x}_{n-1}|_\infty^r \le \frac{\epsilon}{\nu_{n-1}^B} \le \epsilon.$$

If we name C_2 the constant in Lemma 4.6 for $M = 1$ and $F = \gamma$, then, similarly,

$$C_2 \mathcal{N}(S\boldsymbol{x}_{n-1})\nu_{n-1}^B |\Delta \boldsymbol{x}_{n-1}|_\infty^r \le C_2 \nu_{n-1}^{B+1} |\Delta \boldsymbol{x}_{n-1}|_\infty^r \le C_2 \epsilon \eta^{n-1} \le C_2 \epsilon; \tag{5.39}$$

it follows that both (5.22) and (5.23) will be satisfied for $\boldsymbol{x} = \boldsymbol{x}_{n-1}$ provided

$$\epsilon \le \min\left[\frac{1}{4C_2},\ 2\,|\gamma'|_\infty\, \Omega(r,\gamma') C_3^{-r}\right], \tag{5.40}$$

where C_3 is the constant of Proposition 3.1. We shall assume that (5.40) is satisfied in the remainder of the proof. We now apply Lemma 5.6 to $\boldsymbol{x}_{n-1}$ and conclude

$$\nu_n \le \mathcal{N}(S\boldsymbol{x}_{n-1})\left[1 + 8C_2\, \mathcal{N}(S\boldsymbol{x}_{n-1})\nu_{n-1}^B |\Delta \boldsymbol{x}_{n-1}|_\infty^r\right] \le \nu_{n-1}\left[1 + 8C_2\epsilon\eta^{n-1}\right], \tag{5.41}$$

where we have used the same arguments as in (5.39), including for the first time the effect of $\eta < 1$ as well. For ϵ satisfying (5.38, 5.40) we can likewise conclude

$$\nu_j \le \nu_{j-1}\left[1 + 8C_2\epsilon\eta^{j-1}\right], \qquad j = 0, \ldots, n-1.$$

It follows that

$$\nu_n \le \nu_0 \prod_{j=0}^{n-1}\left[1 + 8C_2\epsilon\eta^j\right] \le \nu_0 \prod_{j=0}^{n-1} \exp\left(8C_2\epsilon\eta^j\right) = \nu_0 \exp\left(8C_2\epsilon \sum_{j=0}^{n-1} \eta^j\right) \le \nu_0 e^{8C_2\epsilon/(1-\eta)} \le R,$$

provided

$$\epsilon \le \frac{1-\eta}{8C_2} \ln \frac{R}{\nu_0}, \tag{5.42}$$

as will be assumed henceforth. Again, note that ϵ can be chosen greater than zero because of (5.27, 5.32). This proves (5.34) for $j = n$.

By Lemma 5.3, we also have

$$|\Delta \boldsymbol{x}_n|_\infty = \sup_k \max((\Delta \boldsymbol{x}_n)_{2k},\ (\Delta \boldsymbol{x}_n)_{2k+1}) \le \delta_n \sup_k (\Delta \boldsymbol{x}_{n-1})_k, = \delta_n\, |\Delta \boldsymbol{x}_{n-1}|_\infty\,,$$

where

$$\delta_n = 1 - \frac{1-b}{2 + c'_\gamma\, |\Delta \boldsymbol{x}_{n-1}|_\infty^r}.$$

By using (5.35) we obtain

$$\delta_n \le 1 - \frac{1-b}{2 + c'_\gamma\, |\Delta \boldsymbol{x}_0|_\infty^r} \le 1 - \frac{1-b}{2 + c'_\gamma R^{-B-1}\epsilon} = \frac{1 + b + c'_\gamma R^{-B-1}\epsilon}{2 + c'_\gamma R^{-B-1}\epsilon} \le \frac{1+b}{2} + \frac{1}{2}c'_\gamma R^{-B-1}\epsilon \le \delta,$$

provided

$$\epsilon \le \left(c'_\gamma\right)^{-1} R^{B+1}\left[2\delta - 1 - b\right], \tag{5.43}$$

where the right hand side is positive by (5.32). Assuming (5.43) is satisfied, we have thus

$$|\Delta \boldsymbol{x}_n|_\infty \le \delta\, |\Delta \boldsymbol{x}_{n-1}|_\infty\,,$$

which proves (5.35) for $j = n$. Finally, by also using (5.41, 5.36), we find

$$\nu_n^{B+1} |\Delta \boldsymbol{x}_n|_\infty^r \leq [1 + 8C_2\epsilon]^{B+1} \nu_{n-1}^{B+1} \delta^r |\Delta \boldsymbol{x}_{n-1}|_\infty^r \leq \eta \nu_{n-1}^{B+1} |\Delta \boldsymbol{x}_{n-1}|_\infty^r \leq \epsilon \eta^n,$$

provided

$$\epsilon \leq \frac{1}{8C_2} \left[\left(\eta \delta^{-r}\right)^{\frac{1}{B+1}} - 1 \right]; \tag{5.44}$$

since $\eta > \delta^r$ by (5.32), we can pick $\epsilon > 0$. This completes the proof of (5.36), the last induction step, for $j = n$. It thus suffices to choose $\epsilon > 0$ so that (5.38, 5.40, 5.42, 5.43, 5.44) are satisfied to derive all the results in the theorem. □

Theorem 5.7 relied strongly on $\beta > 1$, since this assumption is needed for Lemmas 5.4 and 5.6. For the special case $S = S_2$, we can prove similar results for all $\beta > 0$, without even imposing bounds on $|\Delta \boldsymbol{x}_0|_\infty$.

Theorem 5.8. *Suppose $S = S_2$, $\beta > 0$ and $\boldsymbol{x}_0$ is strictly increasing. Then $\boldsymbol{x}_j$ is strictly increasing for all $j \geq 0$. If $\gamma \in C^\beta(\mathbb{R})$ with $0 < \beta < 1$, there is a C such that*

$$|\Delta \boldsymbol{x}_j|_\infty \leq \frac{C}{1 + j^{\frac{\beta}{1-\beta}}}, \qquad \forall j \geq 0. \tag{5.45}$$

If $\gamma \in C^\beta(\mathbb{R})$ with $\beta \geq 1$ or $\gamma \in \mathrm{Lip}^1(\mathbb{R})$, there is a $\delta < 1$ such that

$$|\Delta \boldsymbol{x}_j|_\infty \leq \delta^j |\Delta \boldsymbol{x}_0|_\infty, \qquad \forall j \geq 0. \tag{5.46}$$

If $\gamma \in C^\beta(\mathbb{R})$ with $\beta > 1$ and $\mathcal{N}(\boldsymbol{x}_0) < \infty$ the quantity $\mathcal{N}(\boldsymbol{x}_j)$ remains bounded for all j and it satisfies the estimate,

$$\mathcal{N}(\boldsymbol{x}_j) \leq \mathcal{N}(\boldsymbol{x}_0) \exp\left(\frac{z_0(2 + z_0)}{r}\right), \qquad z_0 = \Omega(r, \gamma') |\Delta \boldsymbol{x}_0|_\infty^r, \qquad r = \min(\beta - 1, 1). \tag{5.47}$$

Proof. When $S = S_2$ the help sequences $\boldsymbol{\varepsilon}^x$, $\boldsymbol{\varepsilon}^y$ and $\boldsymbol{\alpha}$ are trivially zero, so we can take $b = 0$ in Lemma 5.3. It follows directly that $\boldsymbol{x}_j$ is strictly increasing for all j since $\boldsymbol{x}_0$ is strictly increasing. Furthermore, if $\beta > 1$,

$$\delta_j = 1 - \frac{1}{2 + c'_\gamma |\Delta \boldsymbol{x}_j|_\infty^r} < 1, \qquad c'_\gamma = \Omega(r, \gamma'), \qquad r = \min(\beta - 1, 1), \tag{5.48}$$

and

$$|\Delta \boldsymbol{x}_{j+1}|_\infty = \sup_k \max((\Delta \boldsymbol{x}_{j+1})_{2k}, (\Delta \boldsymbol{x}_{j+1})_{2k+1}) \leq \sup_k \delta_j (\Delta \boldsymbol{x}_j)_k = \delta_j |\Delta \boldsymbol{x}_j|_\infty, \qquad \forall j. \tag{5.49}$$

This shows that $|\Delta \boldsymbol{x}_j|_\infty$ is decreasing, and, by (5.48), $\delta_j \leq \delta_0$ for all j. We can hence replace δ_j by $\delta_0 < 1$ in (5.49), which proves (5.46). Suppose now that $0 < \beta \leq 1$, and introduce the function,

$$f(x) := x \left(1 - \frac{1}{(2 + c_\gamma / x^{1-\beta})^{1/\beta}}\right), \quad c_\gamma = \Omega(\beta, \gamma),$$

so that by Lemma 5.3 we have

$$\sup_k \max((\Delta \boldsymbol{x}_{j+1})_{2k}, (\Delta \boldsymbol{x}_{j+1})_{2k+1}) \leq f\left((\Delta \boldsymbol{x}_j)_k\right). \tag{5.50}$$

If $\gamma \in \mathrm{Lip}^1$ (and in particular if $\gamma \in C^1$), (5.50) gives

$$\sup_k \max((\Delta \boldsymbol{x}_{j+1})_{2k}, (\Delta \boldsymbol{x}_{j+1})_{2k+1}) \leq (\Delta \boldsymbol{x}_j)_k \left(1 - \frac{1}{(2 + c_\gamma)}\right),$$

with the same constant for all j, and we can take $\delta = 1 - 1/(2 + c_\gamma)$ in (5.46). When $\beta < 1$ we can write $f(x) = \tilde{f}(x^{1-\beta})$ where

$$\tilde{f}(x) := x^{\frac{1}{1-\beta}} \left(1 - g(x)^{1/\beta}\right), \qquad g(x) := \frac{x}{2x + c_\gamma},$$

and we claim that

$$\tilde{f}'(x) \geq 0, \qquad \forall x > 0. \tag{5.51}$$

Since $(1+x)^\beta \leq 1 + \beta x$ for $0 \leq \beta \leq 1$ we have

$$\begin{aligned} \tilde{f}'(x) &= x^{\frac{\beta}{1-\beta}} \left(\frac{1 - g(x)^{1/\beta}}{1-\beta} - \frac{x}{\beta} g(x)^{\frac{1-\beta}{\beta}} g(x)' \right) = \frac{x^{\frac{\beta}{1-\beta}}}{1-\beta} \left(1 - \left[g(x) \left(1 + \frac{(1-\beta) x g'(x)}{\beta g(x)} \right)^\beta \right]^{1/\beta} \right) \\ &\geq \frac{x^{\frac{\beta}{1-\beta}}}{1-\beta} \left(1 - \left[g(x) + (1-\beta) x g'(x) \right]^{1/\beta} \right). \end{aligned}$$

But

$$g(x) + (1-\beta) x g'(x) = \frac{x}{2x + c_\gamma} + \frac{(1-\beta) x c_\gamma}{(2x + c_\gamma)^2} \leq \frac{2x}{x + c_\gamma} + \frac{x c_\gamma}{(2x+c_\gamma)^2} = \frac{1}{2} \left(1 - \left(\frac{c_\gamma}{x + c_\gamma} \right)^2 \right) < 1.$$

This shows (5.51); it immediately follows that likewise $f'(x) \geq 0$ for all $x > 0$. Then, by (5.50),

$$|\Delta \boldsymbol{x}_{j+1}|_\infty \leq \sup_k \, f((\Delta \boldsymbol{x}_j)_k) = f\left(\sup_k \, (\Delta \boldsymbol{x}_j)_k \right) = f\left(|\Delta \boldsymbol{x}_j|_\infty \right), \qquad \forall j \geq 0. \tag{5.52}$$

Proving (5.45) reduces thus to a simple statement about iterating a function from $\mathbb{R}^+$ to itself: if we define

$$h_0 = |\Delta \boldsymbol{x}_0|_\infty, \qquad h_n = f(h_{n-1}),$$

then (5.45) will follow from (5.52) if we can prove that

$$h_n \leq C(1+n)^{-\beta/(1-\beta)}. \tag{5.53}$$

We shall establish (5.53) by induction. Suppose (5.53) holds for $n \leq N-1$. We then have, with the shorthand $\mu := \beta/(1-\beta)$,

$$h_N = f(h_{N-1}) \leq f\left(C N^{-\mu} \right) \leq C N^{-\mu} \left[1 - \left(2 + c_\gamma C^{-(1-\beta)} N^\beta \right)^{-1/\beta} \right]. \tag{5.54}$$

To prove that this is bounded above by $C(1+N)^{-\mu}$, it suffices to show that

$$(1+N)^\mu N^{-\mu} \left[1 - \left(2 + c_\gamma C^{-(1-\beta)} N^\beta \right)^{-1/\beta} \right]$$

is bounded above by 1. This is equivalent to

$$N^\beta \left\{ \left[1 - \left(1 - \frac{1}{N+1} \right)^\mu \right]^{-\beta} - 2 \right\}^{-1} \leq C^{1-\beta} c_\gamma, \tag{5.55}$$

provided the quantity between curly brackets is positive; the latter is the case if

$$N > N_0 = \left\lceil \left[1 - (1 - 2^{-1/\beta})^{1/\mu} \right]^{-1} \right\rceil.$$

Since the left hand side of (5.55) is uniformly bounded for $N > N_0$, the inequality (5.55) is clearly satisfied, uniformly in $N > N_0$, for all C exceeding some threshold value $\hat{C}$. For $n \leq N_0$, we have $h_n \leq h_0$, hence

$$h_n \leq h_0 (1+n)^{-\mu} (1+N_0)^\mu. \tag{5.56}$$

Set now $C = \max[h_0(1+N_0)^\mu, \hat{C}]$. Then h_n satisfies (5.53) for $n \leq N_0$ by (5.56), and for $n > N_0$ by our induction argument, starting from the initial inequality $h_{n_0} \leq C(1+N_0)^{-\mu}$. Since (5.53) thus holds for all n, we have proved (5.45).

For the statement about $\mathcal{N}(\boldsymbol{x}_j)$ we observe that, by (5.10) in Lemma 5.3,

$$\mathcal{N}(\boldsymbol{x}_{j+1}) = \sup_{\substack{k \\ \ell_1=0,1 \\ \ell_2=\pm 1}} \frac{(\Delta \boldsymbol{x}_{j+1})_{2k+\ell_1}}{(\Delta \boldsymbol{x}_{j+1})_{2k+\ell_1+\ell_2}} \le \sup_{\substack{k \\ \ell=-1,0,1}} \frac{\delta_j (\Delta \boldsymbol{x}_j)_k}{(1-\delta_j)(\Delta \boldsymbol{x}_j)_{k+\ell}} \le \frac{\delta_j}{1-\delta_j} \mathcal{N}(\boldsymbol{x}_j), \tag{5.57}$$

where δ_j is given by (5.48). For simplicity, set $z_j = c'_\gamma |\Delta \boldsymbol{x}_j|_\infty^r$. Then, by the result (5.46) above, $\{z_j\}$ is an exponentially decreasing sequence such that $z_j \le z_0 \delta_0^{jr}$. Moreover, induction on (5.57) gives

$$\begin{aligned} \mathcal{N}(\boldsymbol{x}_n) &\le \mathcal{N}(\boldsymbol{x}_0) \prod_{j=0}^{n-1} \frac{\delta_j}{1-\delta_j} = \mathcal{N}(\boldsymbol{x}_0) \prod_{j=0}^{n-1} (1+z_j) \le \mathcal{N}(\boldsymbol{x}_0) \prod_{j=0}^{n-1} \left(1 + z_0 \delta_0^{jr}\right) \\ &\le \mathcal{N}(\boldsymbol{x}_0) \exp\left(\sum_{j=0}^{n-1} z_0 \delta_0^{jr}\right) \le \mathcal{N}(\boldsymbol{x}_0) \exp\left(\frac{z_0}{1-\delta_0^r}\right) \le \mathcal{N}(\boldsymbol{x}_0) \exp\left(\frac{z_0(2+z_0)}{r}\right), \end{aligned}$$

where the last step follows from

$$\frac{1}{1-\delta_0^r} = \frac{1}{1-\left(1-\frac{1}{2+z_0}\right)^r} \le \frac{2+z_0}{r}.$$

This proves (5.47). ☐

Remark: Combining the two theorems with some results shown later on in the paper suggests the following normal multiresolution procedure for all curves Γ, with $\gamma \in C^\beta$, $\beta > 1$. Let S_w be the hybrid scheme $S_w = wS_4 + (1-w)S_2$ with $0 < w \le 1$. In Appendix B it is shown that this scheme is weakly contractive with bound $R_w \in [3+2\sqrt{2}, \infty)$. Starting with an initial strictly increasing sequence $\boldsymbol{x}_0$ for which both $|\Delta \boldsymbol{x}_0|_\infty$ and $\mathcal{N}(\boldsymbol{x}_0)$ may be large, we use S_2 until, for some $w \in (0,1]$,

$$\mathcal{N}(\boldsymbol{x}_j) \le R_w \quad \text{and} \quad R_w^{B+1} |\Delta \boldsymbol{x}_j|_\infty^r \text{ is sufficiently small for } S_w.$$

Since $\mathcal{N}(\boldsymbol{x}_j)$ remains bounded when S_2 is used, by (5.47), and $\lim_{w\to 0} R_w = \infty$, these conditions will be satisfied after a finite number of refinement levels. By Theorem 5.7 one can then use S_w and obtain convergence. Moreover, by (6.23) in Theorem 6.3 the non-uniformity $\mathcal{N}(\boldsymbol{x}_j)$ will converge to one, since $\hat{\sigma} > 1$ for S_w, as shown in Section 7. Therefore, both $\mathcal{N}(\boldsymbol{x}_j)$ and $|\Delta \boldsymbol{x}_j|_\infty$ can be made as small as we like, and after yet another finite number of steps, we can finish off the construction with a weakly contractive scheme of our choice.

6. Regularity, Approximation Quality and Stability

In this section we will consider the regularity of the parameterization, the decay of wavelet coefficients, and the stability of the scheme.

Normal multiresolution induces a parameterization of the curve Γ, as exemplified in Figure 4. Analytically, this parameterization is described as follows: we define, at every level j, $\mathtt{x}_j : [0,1] \mapsto \mathbb{R}$ to be the piecewise affine map with breakpoints at the $t_{j,k} = 2^{-j}k$, $k = 0, \dots, 2^j$, and for which $\mathtt{x}_j(t_{j,k}) = x_{j,k}$, see Figure 5. If $|\Delta \boldsymbol{x}_j|_\infty \to 0$ for $j \to \infty$, then the $\mathtt{x}_j$ converge uniformly to a function $\mathtt{x}(t)$. The parameterization of the curve Γ induced by the normal multiresolution then maps $t \in [0,1]$ to $(\mathtt{x}(t), \gamma(\mathtt{x}(t)))$; we shall call this the *normal parameterization* of the curve Γ, and denote it by Γ as well, $\Gamma(t) := (\mathtt{x}(t), \gamma(\mathtt{x}(t)))$. More generally, $\mathtt{x}_j : [0, N-1] \mapsto \mathbb{R}$ if we start with N points at level 0. The domain of $\mathtt{x}_j$ will be denoted $\mathtt{I}$.

As we discussed in Section 5.1, in general the curve Γ is broken up in several pieces, for some of which the x-coordinate is used as the "basic coordinate", while others use the y-coordinate in this capacity (one then has to make the obvious changes to define the parameterization $\Gamma(t)$). In this case, the parameterizations knit together naturally—they describe the geometric construction of the normal multiresolution, independent of the break-ups we use to prove our theorems. For simplicity, we shall always implicitly assume that we work within one of these pieces; this situation is always attained locally after a finite number of refinement steps. Note that the normal

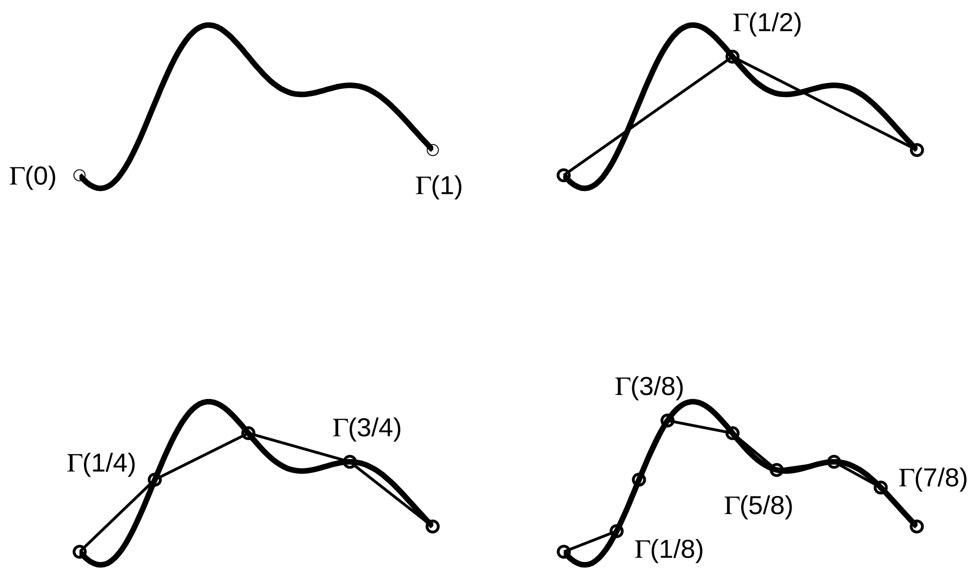

FIGURE 4. Example of how the normal multiresolution induces a parameterization, here with $S = S_2$.

parameterization need not be smooth, even if Γ is. For instance, consider the normal multiresolution as applied to the curve in Figure 6, which consists of a 180° circle arc and a straight, tangent line segment with length equal to the diameter of the circle; for the prediction subdivision scheme we take $S = S_2$. At level zero, we have $(x_{0,0}, y_{0,0}) = (0,1) = \Gamma(0)$ and $(x_{0,1}, y_{0,1}) = (1,0) = \Gamma(1)$. Because of the special construction of Γ, the first inserted point $(x_{1,1}, y_{1,1})$ coincides with the origin, $(0,0) = \Gamma(1/2)$. After that, the normal multiresolution will induce a parameterization that corresponds to arc length for both the right and the left piece of the curve:

$$\Gamma(t) = \begin{cases} \frac{1}{2}(-\sin(2t\pi), 1 + \cos(2t\pi)), & 0 \le t < \frac{1}{2}, \\ (2t-1, 0), & \frac{1}{2} \le t \le 1. \end{cases}$$

However, the two pieces have different lengths, so the parameterization must have a discontinuity in its gradient, indeed

$$\left| \frac{d\Gamma(t)}{dt} \right| = \begin{cases} \pi, & 0 \le t < \frac{1}{2}, \\ 2, & \frac{1}{2} < t \le 1. \end{cases}$$

In this case the curve Γ is C^{2-}, yet its normal parameterization is only Lipschitz. This is because the regularity of the parameterization turns out to be limited not only by the smoothness of the curve, but also by the smoothness of the subdivision scheme, as shown by the following argument. Let $\Gamma(t)$ be the normal parameterization obtained with the two-point scheme as predictor, of a very smooth curve Γ. We then have, by definition,

$$\Big(\Gamma(t+h) - \Gamma(t-h)\Big) \cdot \Big(\Gamma(t+h) - 2\Gamma(t) + \Gamma(t-h)\Big) = 0,$$

at odd dyadic points ($t = (2k+1)2^{-j}$, $h = 2^{-j}$), where ' $\cdot$ ' stands for the $\mathbb{R}^2$ inner product, $(u,v) \cdot (u',v') = uu' + vv'$. Now *if* the parameterization were $C^{4+\varepsilon}$, with $\varepsilon > 0$, then we could Taylor expand $\Gamma(t \pm h)$ around t and obtain

$$\frac{d}{dt}\left|\frac{d\Gamma(t)}{dt}\right|^2 + \frac{h^2}{12}\left(\frac{d^3}{dt^3}\left|\frac{d\Gamma(t)}{dt}\right|^2 - \frac{d}{dt}\left|\frac{d^2\Gamma(t)}{dt^2}\right|^2\right) = \mathrm{O}(h^4),$$

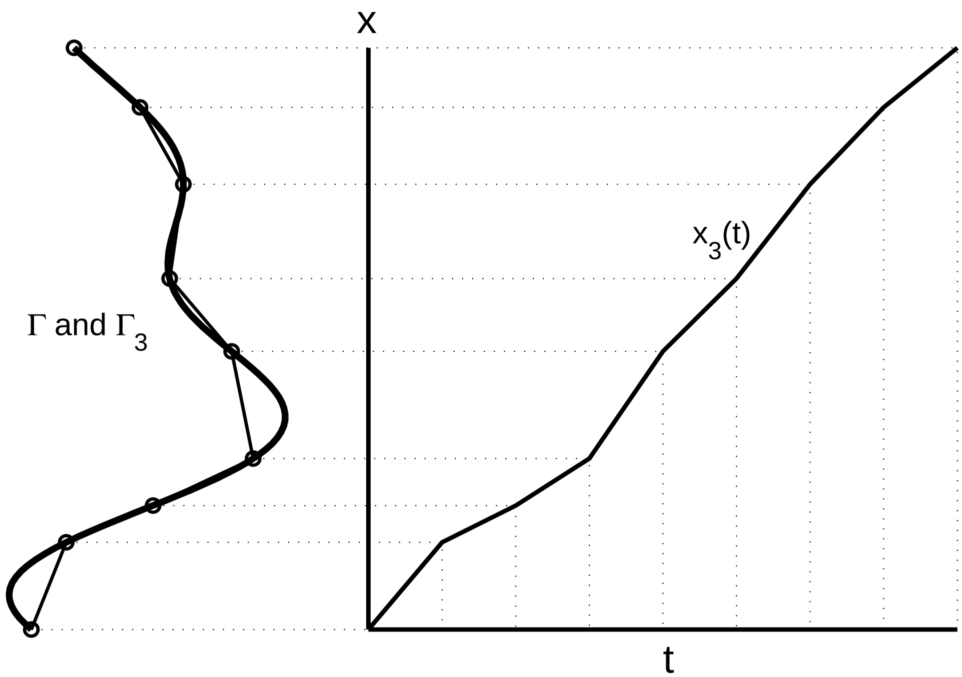

FIGURE 5. The relationship between $\Gamma(t)$, $\Gamma_j(t)$ and $\mathtt{x}_j(t)$, exemplified for $j = 3$. Γ is the same curve as in Figure 4.

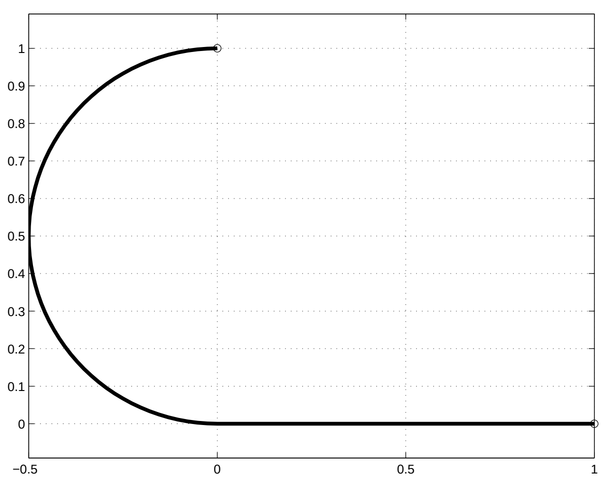

FIGURE 6. Example of a curve with non smooth parameterization for the two-point scheme.

at dyadic points; if $\Gamma(t) \in C^{4+\varepsilon}$, then this equation extends to all t, h. By letting $h \to 0$ we see that we must have $|\Gamma'| = \text{const}$ and $|\Gamma''| = \text{const}$ for this to hold, i.e. the curve Γ must be either a straight line or a circle segment, which is obviously not the case for general smooth curves Γ.

The example in Figure 6 also illustrates an interesting point concerning the link between the decay of the wavelet coefficients and the regularity of the normal parameterization. Although the normal parameterization of the curve in Figure 6 is only Lipschitz where the circle and line segment meet, an application of Theorem 6.3 below shows that the wavelet coefficients $w_{j,k}$ decay uniformly as $2^{-(2^-)j}$; this shows that one cannot hope to derive the wavelet coefficient decay simply from applying Taylor expansion arguments to the normal parameterization.

6.1. **General Assumptions.** We are going to assume that there is a set of strictly increasing sequences $\{\boldsymbol{x}_j\}$ generated by the normal scheme described above, such that $\boldsymbol{x}_{j+1} = N_j \boldsymbol{x}_j$. We let S denote the interpolating predictor operator used in the scheme. As in Section 4.1 we assume it is a linear, stationary, local and bounded subdivision operator. Also, S is characterized by its order $\mathcal{P}$, its smoothness σ and the integer $p = p_{\text{opt}}$. We will strengthen our assumptions in this section and assume that

$$\hat{\sigma} \geq 1. \tag{6.1}$$

(Note that this implies that σ is p_{opt}-suitable if $\sigma = 1$.) As usual, we assume that the curve $\gamma(x)$ has a certain Hölder smoothness, given by the parameter β,

$$\gamma \in C^\beta(\mathbb{R}).$$

Also, let $\mathtt{I} \subset \mathbb{R}$ be a bounded interval of the form $[0, N-1]$ with $N > 1$ an integer; $\mathtt{I}$ is the domain of $\mathtt{x}(t)$.

6.2. **Preliminary Lemmas.** We start out with some technical lemmas, which will allow us to improve upon the $|\Delta \boldsymbol{x}_j|_\infty \leq c\delta^j$ estimate by "bootstrapping." The basic idea is the following: by Lemma 5.5 we can use the smoothness of γ to bound the difference between $\boldsymbol{x}_{j+1}$ and the "predicted" $S\boldsymbol{x}_j$ by the difference between $S\gamma(\boldsymbol{x}_j)$ and $\gamma(S\boldsymbol{x}_j)$; next we can use our commutation estimates from Section 4.4: Theorem 4.7 uses the smoothness of γ again to transform exponential decay of $|\Delta \boldsymbol{x}_j|_\infty$ into exponential decay of $|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$, hence of $|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty$, showing that the $\boldsymbol{x}_j$ are "almost" generated by the subdivision S, in the sense of Section 4; finally, Theorem 4.2 converts this into a new estimate on $|\Delta \boldsymbol{x}_j|_\infty$. The miracle is that in going through these steps, the decay rate of $|\Delta \boldsymbol{x}_j|_\infty$ improves.

Lemma 6.1. *Let $\{\boldsymbol{x}_j\}$, S, $\mathcal{P}$, p, σ, γ and β be given as in Section* 6.1. *Suppose the first differences of $\boldsymbol{x}_j$ converge exponentially to zero,*

$$|\Delta \boldsymbol{x}_j|_\infty \leq C\delta^j, \qquad 0 < \delta < 1, \tag{6.2}$$

and

$$\beta > 1.$$

Then

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \leq \text{const} \begin{cases} j, & \sigma = 1, \;\; p > 1, \\ 1, & \text{otherwise.} \end{cases} \tag{6.3}$$

Proof. When $\delta \leq 1/2$ the result is trivial. Consider therefore $1/2 < \delta < 1$ and set $r = \min(\beta - 1, 1) > 0$. Let $c_0 = c$ and $\kappa_0 = 1 + \log_2 \delta + a$ where $a \geq 0$ is chosen such that $0 < \kappa_0 < 1$ and $(1+r)^n(\log_2 \delta + a) \neq -\sigma$ for all n. Then

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \leq c_0 2^{j\kappa_0}. \tag{6.4}$$

Furthermore, we define the strictly decreasing sequence $\{\kappa_n\}$ by $\kappa_n = 1 - (1+r)^n(1-\kappa_0)$. We claim that if $\kappa_n > -\sigma + 1$, there is a constant c_n for which

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \leq c_n 2^{j\kappa_n}, \qquad \forall j. \tag{6.5}$$

By (6.1) and (6.4) this holds for $n = 0$. Assume that (6.5) is true for $n = \ell$. Since $\beta > 1$ we can use Lemma 5.5 and subsequently Theorem 4.7 with $M = 1$, $\alpha = 0$ and $s = \kappa_\ell$ to get

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \le |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty \le c'_\ell 2^{-j(1+r)(1-\kappa_\ell)} = c'_\ell 2^{-j(1-\kappa_{\ell+1})}.$$

This holds for $j \ge j_0$. Set

$$c''_\ell = c'_\ell + \max_{0 \le j \le j_0} 2^{j(1-\kappa_{\ell+1})} |\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty .$$

Then

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \le c''_\ell 2^{-j(1-\kappa_{\ell+1})}, \qquad \forall j \ge 0.$$

If $\kappa_{\ell+1} > -\sigma + 1$ we can pick a p-suitable μ so that $p - (1 - \kappa_{\ell+1}) > \mu$. Theorem 4.2 with $\nu = 1 - \kappa_{\ell+1}$ and $\alpha = 0$ then applies, yielding

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \le c\left(\left|\boldsymbol{x}_0^{[1]}\right|_\infty + c''_\ell\right) 2^{j[\max(p-\nu,\mu)+1-p]} = c_{\ell+1} 2^{j\kappa_{\ell+1}},$$

where we note that $\left|\boldsymbol{x}_0^{[1]}\right|_\infty = |\Delta\boldsymbol{x}_0|_\infty \le C$ by (6.2). The claim follows by induction. Now let $n > 0$ be the first index such that $\kappa_n < -\sigma + 1 \le 0$. (The case $\kappa_n = -\sigma + 1$ is excluded by the choice of a above.) We still have

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \le c''_{n-1} 2^{-j(1-\kappa_n)}, \qquad \forall j \ge 0.$$

Pick a p-suitable μ so that $\kappa_n < \mu + 1 - p$. We can again apply Theorem 4.2 and obtain

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \le c\left(\left|\boldsymbol{x}_0^{[1]}\right|_\infty + c''_{n-1}\right)\left[1 + j^{\eta_1} 2^{j(1-p+\mu)}\right] \le c'\left[1 + j^{\eta_1} 2^{j(1-p+\mu)}\right].$$

If $\sigma > 1$, then we can choose a p-suitable μ so that $1 - p + \mu = 1 - \sigma + \varepsilon < 0$, and we obtain $\left|\boldsymbol{x}_j^{[1]}\right|_\infty \le \text{const}$. If $\sigma = 1$, then (6.1) allows us to pick $\mu = p - 1$. Since $\kappa_n < 0$ we have $\nu = 1 - \kappa_n > 1$ implying $\mu > p - \nu$. By Theorem 4.2 we then have $\eta_1 = 1$ if $p > 1$ and $\eta_1 = 0$ if $p = 1$. This proves (6.3). □

Remarks:

1. The result of Lemma 6.1 is quite remarkable: even though we started from (6.2) with no other restriction on δ other than $\delta < 1$, the simple restriction (6.1) allows us to "transform" this possibly quite low decay rate into $|\Delta\boldsymbol{x}_j|_\infty \le C2^{-j}$ (with an extra factor polynomial in j if $p > 1$ and $\sigma = 1$).
2. If $\sigma < 1$ a similar argument proves that the decay $|\Delta\boldsymbol{x}_j|_\infty \le C\delta^j$, with $2^{-\sigma} < \delta < 1$, implies the stronger decay $|\Delta\boldsymbol{x}_j|_\infty \le C2^{-j(\sigma-\varepsilon)}$ for all $\varepsilon > 0$.

The next lemma shows that a similar bootstrapping works for higher order divided differences. If λ equals either a real number r or a "generalized number" r^-, we shall use the convention that $\lambda^- = r^-$. The notation r^- used here was defined in Section 3.1.

Lemma 6.2. *Let $\{\boldsymbol{x}_j\}$, S, $\mathcal{P}$, p, σ, $\hat\sigma$, γ and β be given as in Section* 6.1. *Suppose we have the following bounds on the first q divided differences of $\boldsymbol{x}_j$,*

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \le \text{const}\begin{cases} 1, & 1 \le m < q, \\ j^\alpha, & m = q, \end{cases} \qquad j > 0. \tag{6.6}$$

If

$$2 \le p \le \mathcal{P}, \qquad 1 \le q < \min(\hat\sigma, \beta) =: Q, \tag{6.7}$$

then we get bounds of the first $q+1$ divided differences of $\boldsymbol{x}_j$,

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \le \text{const}\begin{cases} 1, & 1 \le m < q+1, \\ j^\eta 2^{j(1-\kappa)}, & m = q+1, \end{cases} \tag{6.8}$$

where κ and η are given by

$$\kappa = \min(1,\ \beta - q,\ \hat\sigma - q) = \min(1, Q - q), \tag{6.9}$$

$$\eta = \begin{cases} 0, & \kappa = \sigma^- - q, \\ \eta', & \text{otherwise}, \end{cases} \tag{6.10}$$

with

$$\eta' = \begin{cases} 1, & \sigma = \min(\beta, q+1), \\ 0, & \text{otherwise}, \end{cases} + \begin{cases} 1, & \kappa = 1,\ p > q+1, \\ 0, & \text{otherwise}. \end{cases} \tag{6.11}$$

Proof. Since $\beta > 1$, by (6.7), we can apply Lemma 5.5, which gives

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \le c\, |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty .$$

(Strictly speaking Lemma 5.5 gives $|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \le |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$ for j exceeding some j_0; by adjusting the constant we obtain the inequality for all j.) By Theorem 4.7 (with $M = q$, $r = \min(\beta - q, 1) > 0$, $s = 0$, and the same α),

$$|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty \le c j^{\alpha'} 2^{-j(q+r)}, \qquad \alpha' = \begin{cases} \alpha(q+r), & q = 1, \\ 0, & 0 < 1 - r < q - 1, \\ \alpha, & r = 1, q > 1. \end{cases} \tag{6.12}$$

Pick μ to be p-suitable, $\mu = p - \sigma + \varepsilon$ with $\varepsilon > 0$; if $p - \sigma$ is p-suitable we can pick $\varepsilon = 0$. Theorem 4.2 can now be applied to the sequences $\boldsymbol{x}_j$ with $\nu = q + r$, $\alpha = \alpha'$ and

$$\varrho = \max(p - q - r, \mu) = p - q + \max(-r, q - \sigma + \varepsilon) = p - q - \min(1, \beta - q, \sigma - q - \varepsilon) =: p - q - \lambda.$$

We get

$$\begin{aligned} \left|\boldsymbol{x}_j^{[m]}\right|_\infty &\le C\left(\left|\boldsymbol{x}_0^{[m]}\right|_\infty + c\right)\left(1 + j^{\eta_m} 2^{j(m-q-\lambda)}\right) \le C'\left(\left|\boldsymbol{x}_0^{[1]}\right|_\infty + 1\right)\left(1 + j^{\eta_m} 2^{j(m-q-\lambda)}\right) \\ &\le C''\left(1 + j^{\eta_m} 2^{j(m-q-\lambda)}\right), \end{aligned} \tag{6.13}$$

for $0 \le m \le p$. By (6.7), we can choose ε so that $\lambda > 0$. For $1 \le m \le q$, it follows that the second term in (6.13) decays to zero as $j \to \infty$, so that $\left|\boldsymbol{x}_j^{[m]}\right|_\infty \le \text{const}$ for $1 \le m \le q$. This means we get (6.6) with $\alpha = 0$ and we can bootstrap the arguments used so far to obtain $\alpha' = 0$ in (6.12). We have $q + 1 \le p$ because of (6.7) and $\sigma \le p$; (6.8, 6.10) follows from (6.13) and (4.8) with $m = q + 1$. □

Remark: Lemma 6.1 showed that we could bound $\left|\boldsymbol{x}_j^{[1]}\right|_\infty$ polynomially in j; Lemma 6.2 can then be used to prove by induction that for all $m < \min(\sigma, \beta)$, we have $\left|\boldsymbol{x}_j^{[m]}\right|_\infty \le \text{const}$; moreover, for $m = \lceil \min(\sigma, \beta) \rceil$, we obtain $\left|\boldsymbol{x}_j^{[m]}\right|_\infty \le c j^\eta 2^{j(1-\kappa)}$, with η and κ given in (6.9, 6.10). In particular, we have bootstrapped a polynomial bound for $\left|\boldsymbol{x}_j^{[1]}\right|_\infty$ into a constant bound, since under our assumptions, $\min(\sigma, \beta) > 1$.

6.3. **Main Result.** Before stating the main result we introduce the "wavelet" coefficients of the normal mesh. At level j, the sequence $\boldsymbol{w}_j$ is defined by

$$w_{j,k} = \left(\left[(\boldsymbol{x}_{j+1} - \boldsymbol{x}^*_{j+1})^2 + (\boldsymbol{y}_{j+1} - \boldsymbol{y}^*_{j+1})^2\right]^{1/2}\right)_{2k+1}. \tag{6.14}$$

Note that the even-indexed elements of $(\boldsymbol{x}_{j+1} - \boldsymbol{x}^*_{j+1})^2 + (\boldsymbol{y}_{j+1} - \boldsymbol{y}^*_{j+1})^2$ are zero. The sequence $\boldsymbol{w}_j$ measures the quality of the normal multiresolution approximation, in the sense that it compares the "true" curve Γ with the auxiliary curve Γ^*_j that would be constructed using only the sequence $(\boldsymbol{x}_j, \gamma(\boldsymbol{x}_j))$ followed by subdivision using S. Good decay for $\boldsymbol{w}_j$ thus means that the normal multiresolution produces high quality approximation. From another point of view, $\boldsymbol{w}_j$, combined with a sequence of sign bits, contains the information necessary to obtain $\boldsymbol{x}_{j+1}$ from $\boldsymbol{x}_j$. If we wish to compress the total information contained in $\boldsymbol{x}_{j_1}$, or equivalently in the sequences $\boldsymbol{x}_{j_0}, \boldsymbol{w}_{j_0}, \boldsymbol{w}_{j_0+1}, \ldots, \boldsymbol{w}_{j_1-1}$, (where $j_0 < j_1$), then we can try to do this by setting small $w_{j,k}$ to zero. (In order to justify this, we also have to discuss the stability of such a procedure—see below.) If the $|\boldsymbol{w}_j|_\infty$ decay rapidly for smooth curves, then we expect that for piecewise smooth curves, the $w_{j,k}$ pertaining to smooth pieces will be very small for large j, holding promise for effective compression in these regions. All this motivates the following theorem, which estimates the decay of $\boldsymbol{w}_j$ in terms of the smoothness β of the curve as well as the smoothness σ of the subdivision scheme. It also shows the existence of the limiting parameterization and its Hölder regularity.

Theorem 6.3. *Let $\{\boldsymbol{x}_j\}$, S, $\mathcal{P}$, p, σ, γ, β and $\mathtt{I}$ be given as in Section 6.1. Furthermore, let $\mathtt{x}_j(t)$ be a piecewise linear function interpolating the points $\{x_{j,k}\}$ at $t = k2^{-j} \in \mathtt{I}$. Set*

$$Q := \min(\hat{\sigma}, \beta), \qquad Q =: P + \kappa, \qquad P \in \mathbb{N}, \qquad 0 < \kappa \leq 1, \tag{6.15}$$

If the first differences of $\boldsymbol{x}_j$ converge exponentially to zero,

$$|\Delta \boldsymbol{x}_j|_\infty \leq C\delta^j, \qquad 0 < \delta < 1, \tag{6.16}$$

and

$$\beta > 1, \tag{6.17}$$

then the parameterization $\mathtt{x}_j(t)$ converges uniformly exponentially to $\mathtt{x}(t) \in C^{Q^-}(\mathtt{I})$, and $\mathtt{x}^{(P)}(t)$ satisfies the Hölder estimate

$$\left|\mathtt{x}^{(P)}(t + \Delta t) - \mathtt{x}^{(P)}(t)\right| \leq c|\Delta t(1 + |^\kappa| \log|\Delta t||)^\eta, \qquad \forall t,\ t + \Delta t \in \mathtt{I}, \tag{6.18}$$

where

$$\eta = \begin{cases} 0, & Q = \sigma^-, \\ \eta^*, & \text{otherwise}, \end{cases} \tag{6.19}$$

with

$$\eta^* = \begin{cases} 1, & \sigma = \min(\mathcal{P}, \beta) \text{ and } P \geq 1, \\ 0, & \text{otherwise}, \end{cases} \quad + \quad \begin{cases} 1, & Q \in \{1, \dots, p-1\},\ p \geq 2, \\ 0, & \text{otherwise}. \end{cases} \tag{6.20}$$

When $P = 0$, the related wavelet coefficients in (6.14) satisfy

$$|\boldsymbol{w}_j|_\infty \leq cj^{\eta \min(\beta,2)} 2^{-j\min(\beta,2)}, \tag{6.21}$$

and for $P \geq 1$

$$|\boldsymbol{w}_j|_\infty \leq cj^{\eta'} 2^{-j\min(Q+1,\beta,\mathcal{P})}, \qquad \eta' = \begin{cases} 0, & \min(\mathcal{P}, \beta) < Q + 1, \\ \eta, & \min(\mathcal{P}, \beta) \geq Q + 1. \end{cases} \tag{6.22}$$

If $Q > 1$, then for sufficiently large j,

$$\mathcal{N}(\boldsymbol{x}_j) \leq 1 + c2^{-j\theta}, \qquad \theta = \min(Q^- - 1, 1). \tag{6.23}$$

If $Q > 1$ or $S = S_2$, then for sufficiently large j,

$$\sup_k \left|\frac{x_{j+1,2k+1} - x_{j,k}}{x_{j,k+1} - x_{j,k}} - \frac{1}{2}\right| \leq c2^{-j\theta}, \qquad \theta = \begin{cases} \min(Q^- - 1, 1), & Q > 1, \\ \min(\beta - 1, 1), & S = S_2. \end{cases} \tag{6.24}$$

Remarks:

1. This is the same regularity that we get for the limit function of the predictor subdivision scheme when we use the same method of proof. If we take the very special case $\gamma(x) = 1$ for all x, then the normal multiresolution scheme gives $\boldsymbol{x}_{j+1} = S\boldsymbol{x}_j$. In this case $\boldsymbol{w}_j = 0$, and we no longer have a curve approximation problem. However, we can define $\mathtt{x}_j(t)$ as before, and the convergence of $\mathtt{x}_j(t)$ and its derivatives still holds, as a special case of this theorem. Theorem 6.3 can thus be viewed as an extension, without loss in the strength of the estimates, of standard convergence results for linear subdivision such as Theorem 3.2, see e.g. [1, 3].

2. In some cases, we obtain $\mathtt{x}(t) \in \mathrm{Lip}^Q$, which is slightly stronger than stated in the theorem. This happens, e.g. if either $\hat{\sigma} = \sigma$ or $\beta < \sigma$, so that $Q = \min(\sigma, \beta)$, and if in addition $Q \notin \mathbb{Z}$, $P \geq 1$ and $\mathcal{P} > \sigma \neq \beta$; under these conditions η as defined by (6.19, 6.20), equals zero. We leave the details of the other cases where $\mathtt{x}(t) \in \mathrm{Lip}^Q$ to the reader.
3. The statement of Theorem 6.3 may seem overly complicated because of all the different cases, depending on the relative values of $\hat{\sigma}$, β and $\mathcal{P}$. By looking at a few extreme cases, we can get some insight in what is happening. Suppose the curve Γ is very smooth (i.e. β large), whereas S is a reasonable but not very fancy subdivision scheme, so that $\hat{\sigma} \leq p_{\mathrm{opt}} \leq \mathcal{P} < \beta$, implying $Q = \hat{\sigma}$. We then obtain $\mathtt{x} \in C^{\sigma^-}$, i.e. the smoothness of the normal parameterization is that of the subdivision scheme. If $\mathcal{P} \geq \hat{\sigma} + 1$, then the decay of the wavelet coefficients is given by $|\boldsymbol{w}_j|_\infty \leq C j^\eta 2^{-j(\hat{\sigma}+1)}$, i.e. we have a "gain of 1" in this decay rate, when compared to the smoothness of S (but we do not get the full decay rate $\mathcal{P}$ if $\mathcal{P} > \hat{\sigma} + 1$, unlike standard linear wavelet transforms). If we look at the other extreme case, where $\mathcal{P}$ and σ are strictly larger than β, i.e. the subdivision scheme is "smoother" than Γ, then $Q = \beta$, $\mathtt{x} \in C^{\beta^-}$ and $|\boldsymbol{w}_j|_\infty \leq C 2^{-j\beta}$: the decay rate of $|\boldsymbol{w}_j|_\infty$ and the smoothness of $\mathtt{x}(t)$ match and are completely set by the smoothness β of Γ.

Proof of Theorem 6.3. We divide this proof into three parts. In the first part of the proof we show the regularity of $\mathtt{x}(t)$ and (6.18, 6.19, 6.20). In the second part we show the decay estimates for the wavelet coefficients, (6.21, 6.22) and in the last part, we show the remaining statements (6.23, 6.24).

Before starting, we establish the simple inequalities

$$P < Q \leq P + 1 \leq p \leq \mathcal{P}, \qquad 1 \leq Q \leq \beta, \tag{6.25}$$

which follow directly from the definition of these quantities in (6.15) and from (6.1, 6.17). These inequalities will be used extensively below.

Part 1: Regularity of $\mathtt{x}(t)$. The strategy for this part is to show that the sequences $\boldsymbol{x}_j$ are approximately generated by the predictor subdivision scheme S itself, hence that an estimate of the type (4.1) holds for large enough ν. Theorem 4.4 can then be applied to prove convergence and regularity of the limiting function $\mathtt{x}(t)$.

Now, by (6.16, 6.17) we get from Lemma 5.5 that

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty \leq |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty\,, \tag{6.26}$$

for sufficiently large j. We thus only need to bound $|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$ to obtain the (4.1) estimate. Such bounds are given by Theorem 4.7, provided we can control the divided differences of $\boldsymbol{x}_j$. We therefore start by showing that

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \leq \text{const} \begin{cases} 1, & 1 \leq m \leq P, \\ j^{\alpha''} 2^{j(1-\kappa)}, & m = P+1, \end{cases} \tag{6.27}$$

for some integer $\alpha'' \geq 0$ and κ as in (6.15).

The assumptions (6.16, 6.17) are stronger than those of Lemma 6.1, which gives

$$\left|\boldsymbol{x}_j^{[1]}\right|_\infty \leq c\, j^{\alpha'}, \qquad \alpha' = \begin{cases} 1, & \sigma = 1,\ p > 1, \\ 0, & \text{otherwise.} \end{cases} \tag{6.28}$$

This shows (6.27) when $P = 0$. If $P \geq 1$, then we shall prove (6.27) by induction and (6.28) will be our initial step. For our induction step we suppose we have, for some q with $1 \leq q \leq Q - 1$, that, for some $\tilde{\alpha}$,

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \leq \text{const} \begin{cases} 1, & 1 \leq m < q, \\ j^{\tilde{\alpha}}, & m = q. \end{cases} \tag{6.29}$$

The inequality (6.28) shows that this holds for $q = 1$. We then want to derive that the same is true when we replace q by $q + 1$. By Lemma 6.2, this is true for $q \leq P - 1 < Q - 1$, since for these q we have $\min(1, Q - q) = 1$. The induction process thus proves (6.29) for $q = P$. We can now apply Lemma 6.2 one more time, since $P < Q$, which establishes (6.27).

We now proceed to apply Theorem 4.7 to the bound (6.27) and then use Theorem 4.4 to show (6.18, 6.19, 6.20). We divide the arguments into two cases: one where we can use $M = P + 1$ in Theorem 4.7 and one where we have to make do with $M = P$:

Case 1: $P = 0$ **or** $\min(\mathcal{P}, \beta) > P + 1$**.**
This case is chosen such that Theorem 4.7 can be used with $M = P + 1$. The other parameters in Theorem 4.7 are $r = \min(\beta - M, 1) = \min(\beta - P - 1, 1) > 0$, $\alpha = \alpha''$ and $s = 1 - \kappa$, as given in (6.27). We get

$$|\gamma(S\boldsymbol{x}_j) - S\gamma(\boldsymbol{x}_j)|_\infty \leq c j^{\alpha'''} 2^{-j(P+1+\min(\kappa,r))}, \qquad \alpha''' = \begin{cases} \alpha''(1+r), & P = 0, \\ 0, & P > 0, r < \kappa, \\ \alpha'', & P > 0, r \geq \kappa, \end{cases} \tag{6.30}$$

where we used the fact that $\kappa = Q - P = 1$ for $P = 0$ by (6.25). Finally, we combine (6.26) and (6.30) and apply Theorem 4.4 with $\nu = P + 1 + \min(\kappa, r)$ and $\alpha = \alpha'''$. We note that in this case $\beta > P + 1$ so by (6.25) we have $\beta > Q$ and consequently, $Q = \hat{\sigma}$. Since $\nu > P + 1 \geq Q + 1$, we thus have $\nu > \sigma$. Therefore Q, P and κ in (4.15) coincide with (6.15). Let $\{\tilde{\boldsymbol{x}}_j\}$ be the (finite) subsequences of $\{\boldsymbol{x}_j\}$ containing only elements $\tilde{x}_{j,k}$ for which $k2^{-j} \in \mathtt{I}$. Clearly $|\tilde{\boldsymbol{x}}_0|_\infty$ is bounded. Corollary 4.5 applied to $\{\tilde{\boldsymbol{x}}_j\}$ then shows that $\mathtt{x} \in C^{\hat{\sigma}^-}(\mathtt{I}) = C^{Q^-}(\mathtt{I})$. Moreover, (6.18) is given by (4.12) with κ as in (6.15) and η as in (4.13), i.e.

$$\eta = \begin{cases} 1, & Q \in \{1, \ldots, p-1\},\ p \geq 2, \\ 0, & \text{otherwise.} \end{cases} \tag{6.31}$$

What is more, when $P \geq 1$,

$$\min(\mathcal{P}, \beta) > P + 1 \geq Q = \hat{\sigma},$$

so, if $Q = \sigma$, the first term in (6.20) is zero, and (6.31) agrees with (6.19, 6.20).

Case 2: $P \geq 1$ **and** $\min(\mathcal{P}, \beta) \leq P + 1$**.**
In this case we will apply Theorem 4.7 with $M = P$. This is allowed since, by (6.25) we have $M = P < Q \leq \min(\mathcal{P}, \beta)$. We use only the first P bounds in (6.27), i.e.,

$$\left|\boldsymbol{x}_j^{[m]}\right|_\infty \leq \text{const}, \qquad 1 \leq m \leq P.$$

Theorem 4.7, with $r = \min(\beta - M, 1) = \min(\beta - P, 1)$ and $\alpha = s = 0$ then gives

$$|\gamma(S\boldsymbol{x}_j) - S\gamma(\boldsymbol{x}_j)|_\infty \leq c 2^{-j(P+r)} = c 2^{-j\min(\beta, P+1)}. \tag{6.32}$$

We can now apply Corollary 4.5 with $\alpha = 0$ and $\nu = \min(\beta, P + 1)$. We have by (6.25)

$$\min(\hat{\sigma}, \nu) = \min(\hat{\sigma}, \beta, P + 1) = \min(Q, P + 1) = Q,$$

so (4.15) agrees with (6.15). Therefore we obtain again, as in the first case, $\mathtt{x} \in C^{Q^-}(\mathtt{I})$. Finally, (4.16) gives

$$\eta = \begin{cases} 1, & \sigma = \min(\beta, P+1), \\ 0, & \sigma \neq \min(\beta, P+1), \end{cases} + \begin{cases} 1, & Q \in \{1, \ldots, p-1\},\ p \geq 2, \\ 0, & \text{otherwise,} \end{cases} \tag{6.33}$$

and (6.19, 6.20) follows from the restrictions on P, $\mathcal{P}$ and β for this case and (6.25), since

$$\min(\beta, P + 1) = \min(\beta, P + 1, \mathcal{P}) = \min(\beta, \mathcal{P}). \tag{6.34}$$

This completes the first part of the theorem.

Part 2: Decay of the wavelet coefficients. In this second part of the theorem we estimate the wavelet coefficients $\boldsymbol{w}_j$ and show (6.21, 6.22). Again, it is the difference $|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$ that plays the most important role. Indeed, by using Lemma 5.5 we get

$$\begin{aligned}
|\boldsymbol{w}_{j+1}|_\infty &\le |S\boldsymbol{x}_j - \boldsymbol{x}_{j+1}|_\infty + |S\gamma(\boldsymbol{x}_j) - \gamma(\boldsymbol{x}_{j+1})|_\infty \\
&\le |S\boldsymbol{x}_j - \boldsymbol{x}_{j+1}|_\infty + |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty + |\gamma(S\boldsymbol{x}_j) - \gamma(\boldsymbol{x}_{j+1})|_\infty \\
&\le |S\boldsymbol{x}_j - \boldsymbol{x}_{j+1}|_\infty + |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty + |\gamma'|_\infty |S\boldsymbol{x}_j - \boldsymbol{x}_{j+1}|_\infty \\
&\le c\,|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty . \qquad (6.35)
\end{aligned}$$

As in the first part we divide the proof into (the same) two cases, and use the estimates (6.30, 6.32), which were already derived in that part, to bound $|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$ and hence $|\boldsymbol{w}_{j+1}|_\infty$.

Case 1: $P = 0$ **or** $\min(\mathcal{P}, \beta) > P + 1$**.**
In this case we start from the estimate (6.30), and note that by (4.17) we can in fact take $\alpha'' = \eta$ in (6.27), and consequently also in (6.30). Suppose first that $P = 0$. Then, as noted in part one of the proof, $Q = \kappa = 1$. Therefore, still with $r = \min(\beta - P - 1, 1)$,

$$P + 1 + \min(\kappa, r) = 1 + \min(1, r) = 1 + r = 1 + \min(\beta - 1, 1) = \min(\beta, 2),$$

and (6.21) follows from (6.30, 6.35). Assume now instead that $P \ge 1$. Then, by the second restriction in this case (on $\mathcal{P}$, β and P) and (6.25), we must have

$$Q \le P + 1 < \min(\mathcal{P}, \beta) \le \mathcal{P} \quad \Rightarrow \quad Q + 1 \le P + 2 \le \mathcal{P}.$$

Hence,

$$\min(Q+1, \beta, \mathcal{P}) = \min(Q+1, \beta) = \min(P+1+\kappa, \beta, P+2) = P+1+\min(\kappa, \beta-P-1, 1) = P+1+\min(\kappa, r).$$

Furthermore,

$$r = \min(\beta - P - 1, 1) = \min(\beta - Q - 1, 1 - \kappa) + \kappa.$$

Recalling that $0 < \kappa \le 1$, this implies that $r < \kappa$ if and only if $\beta < Q + 1$. But, since $Q + 1 \le \mathcal{P}$, in fact $r < \kappa$ if and only if $\min(\beta, \mathcal{P}) < Q + 1$. Therefore, (6.22) follows from (6.30, 6.35).

Case 2: $P \ge 1$ **and** $\min(\mathcal{P}, \beta) \le P + 1$**.**
By (6.34, 6.25),

$$\min(Q + 1, \beta, \mathcal{P}) = \min(Q + 1, P + 1, \beta) = \min(P + 1, \beta),$$

in this case. Also, by the restrictions on P, $\mathcal{P}$ and β in this case, $\min(\mathcal{P}, \beta) \le P + 1 < Q + 1$. Therefore, $\eta' = 0$ in (6.22), so that (6.22) follows from (6.32, 6.35).

Part 3: Limiting uniformity. For the remaining statements, suppose first that $Q > 1$. Set $\Delta t = 2^{-j}$ and $t = k\Delta t$. Then,

$$\left|\frac{(\Delta \boldsymbol{x}_j)_k}{(\Delta \boldsymbol{x}_j)_{k-1}} - 1\right| = \left|\frac{\mathrm{x}(t + \Delta t) - \mathrm{x}(t)}{\mathrm{x}(t) - \mathrm{x}(t - \Delta t)} - 1\right| \le \left|\frac{c(\Delta t)^{1+\theta}}{\Delta t \mathrm{x}'(t) - c'(\Delta t)^{1+\theta}}\right| \le c(\Delta t)^\theta$$

if Δt is sufficiently small, or, equivalently of j is sufficiently large. It follows that the quantities $a_{j,k}$ defined by

$$a_{j,k} := \max\left(\frac{(\Delta \boldsymbol{x}_j)_k}{(\Delta \boldsymbol{x}_j)_{k+1}}, \frac{(\Delta \boldsymbol{x}_j)_{k+1}}{(\Delta \boldsymbol{x}_j)_k}\right)$$

satisfy the inequality

$$|a_{j,k} - 1| \le \max\left(c(\Delta t)^\theta, \frac{c(\Delta t)^\theta,}{1 - c(\Delta t)^\theta}\right) \le c'(\Delta t)^\theta;$$

now (6.23) follows since $\mathcal{N}(\boldsymbol{x}_j) = \sup_k a_{j,k}$. The same argument proves (6.24) for this case, since

$$\left|\frac{x_{j+1,2k+1} - x_{j,k}}{x_{j,k+1} - x_{j,k}} - \frac{1}{2}\right| = \frac{1}{2}\left|\frac{a_{j,k} - 1}{a_{j,k} + 1}\right| \le \frac{c(\Delta t)^\theta,}{1 - c(\Delta t)^\theta} \le c'(\Delta t)^\theta.$$

Suppose finally that $S = S_2$. For this case $Q = \sigma = 1$ and $p_{\text{opt}} = 1$ (see below). The first part of the theorem then shows that $\mathtt{x}(t) \in \text{Lip}^1$. Moreover, we can take $b = 0$ in Lemma 5.3 and with $c'_\gamma = \Omega(\beta - 1, \gamma')$,

$$\left| \frac{x_{j+1,2k+1} - x_{j,k}}{x_{j,k+1} - x_{j,k}} - \frac{1}{2} \right| = \frac{\max((\Delta \boldsymbol{x}_{j+1})_{2k}, (\Delta \boldsymbol{x}_{j+1})_{2k+1})}{(\Delta \boldsymbol{x}_j)_k} - \frac{1}{2} \leq \delta_j - \frac{1}{2} = \frac{c'_\gamma \left|\Delta \boldsymbol{x}_j\right|_\infty^\theta}{2(2 + c'_\gamma \left|\Delta \boldsymbol{x}_j\right|_\infty^\theta)} \leq c'_\gamma \left|\Delta \boldsymbol{x}_j\right|_\infty^\theta .$$

We get (6.24) for this case by noting that $|\Delta \boldsymbol{x}_j|_\infty \leq \text{const}\ 2^{-j}$ since $\mathtt{x}(t)$ is Lipschitz. □

Remark: We can weaken the assumptions on the predictor subdivision scheme further. It is enough that we use a, possibly nonlinear, interpolating scheme $\tilde{S}$ such that there exist a linear subdivision scheme S with the characteristics stated in the theorem, for which

$$\left| S\boldsymbol{x}_j - \tilde{S}\boldsymbol{x}_j \right|_\infty \leq c2^{-j\lambda},$$

with λ large enough ($\lambda > P + 1 + \min(\kappa, r)$ in case 1 and $\lambda > \min(\beta, P + 1)$ in case 2). Then, (4.1) will still hold for all the cases we care about ($\nu < \lambda$), since

$$\left| \boldsymbol{x}_{j+1} - \tilde{S}\boldsymbol{x}_j \right|_\infty \leq |\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty + c2^{-j\lambda} \leq c' 2^{-j\nu},$$

and similarly with Lemma 5.5 and (6.35).

6.4. **Stability.** In the preceding subsection we established, under certain conditions, a rate of decay for the wavelet coefficients, in the expectation that fast decay will be associated with significant compression without great loss of accuracy. In standard (linear) wavelet decompositions, compression can be achieved by thresholding, which simply discards coefficients along those basis directions that give only small contributions. In the case of (inherently nonlinear) normal multiresolution approximation, it is much less clear a priori that the effect of thresholding (or any other approximation) of the wavelet coefficients can be kept under control. This motivates the stability analysis below, where we investigate the effects of inaccuracies in the initial, coarse scale data and/or the wavelet coefficients on the reconstruction of the curve.

In order to analyze the stability of the normal scheme we set $\boldsymbol{f}_j = (\boldsymbol{x}_j, \boldsymbol{y}_j) \in X^2$. For an element $\boldsymbol{r} = (\boldsymbol{x}, \boldsymbol{y}) \in X^2$ we define

$$|\boldsymbol{r}|_2 = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2} \in X.$$

Then,

$$f_{j+1,2k+1} = (S\boldsymbol{f}_j)_{2k+1} + (\varsigma_j \boldsymbol{w}_j \boldsymbol{n}_j)_k, \qquad f_{j+1,2k} = f_{j,k},$$

where $\boldsymbol{n}_j \in X^2$, $\varsigma_j \in X(\{-1, 1\})$ are the sequences of normal vectors respectively signs defined by

$$\boldsymbol{n}_j = \frac{\Delta \boldsymbol{f}_j^\perp}{|\Delta \boldsymbol{f}_j|_2} := \frac{\left(-\Delta \boldsymbol{y}_j,\ \Delta \boldsymbol{x}_j\right)}{|\Delta \boldsymbol{f}_j|_2}, \qquad \varsigma_{j,k} = \text{sign}\left[(f_{j+1,2k+1} - (S\boldsymbol{f}_j)_{2k+1}) \ \cdot\ \left(-\Delta \boldsymbol{y}_j,\ \Delta \boldsymbol{x}_j\right)\right]$$

Introduce the perturbed sequences $\tilde{\boldsymbol{w}}_j$ and $\tilde{\boldsymbol{f}}_j = (\tilde{\boldsymbol{x}}_j, \tilde{\boldsymbol{y}}_j)$ constructed from $\tilde{\boldsymbol{f}}_0$ by the rule

$$\tilde{f}_{j+1,2k+1} = (S\tilde{\boldsymbol{f}}_j)_{2k+1} + (\varsigma_j \tilde{\boldsymbol{w}}_j \tilde{\boldsymbol{n}}_j)_k, \qquad \tilde{f}_{j+1,2k} = \tilde{f}_{j,k},$$

and similarly,

$$\tilde{\boldsymbol{n}}_j = \frac{\Delta \tilde{\boldsymbol{f}}_j^\perp}{|\Delta \tilde{\boldsymbol{f}}_j|_2} := \frac{\left(-\Delta \tilde{\boldsymbol{y}}_j,\ \Delta \tilde{\boldsymbol{x}}_j\right)}{|\Delta \tilde{\boldsymbol{f}}_j|_2}.$$

We want to show that if $\tilde{\boldsymbol{f}}_0$ is close to $\boldsymbol{f}_0$ and if $|\boldsymbol{w}_j - \tilde{\boldsymbol{w}}_j|_\infty$ is small, then the resulting sequence $\tilde{\boldsymbol{f}}_j$ remains close to $\boldsymbol{f}_j$ for all j.

Theorem 6.4. *Let* $\{\boldsymbol{x}_j\}$, S, $\mathcal{P}$, p, σ, γ *and* β *be given as in Section* 6.1. *Suppose*

$$\left|\boldsymbol{f}_0 - \tilde{\boldsymbol{f}}_0\right|_{2,\infty} \le E_f, \qquad |\boldsymbol{w}_j - \tilde{\boldsymbol{w}}_j|_\infty \le E_w 2^{-js}, \qquad s \ge 0.$$

If the first differences of $\boldsymbol{x}_j$ *converge exponentially to zero,*

$$|\Delta \boldsymbol{x}_j|_\infty \le C\delta^j, \qquad 0 < \delta < 1,$$

and

$$\beta > 1, \qquad \sup_j \mathcal{N}(\boldsymbol{x}_j) < \infty, \qquad \begin{cases} \hat{\sigma} > 1, & p > 1, \\ \hat{\sigma} = 1, & p = 1, \end{cases} \tag{6.36}$$

then there is a constant C *independent of* j, E_f *and* E_w *such that for* $j > 0$,

$$\left|\boldsymbol{f}_j - \tilde{\boldsymbol{f}}_j\right|_{2,\infty} \le C(E_f + E_w)j^\eta, \qquad \eta = \begin{cases} 0, & s > 0, \\ 1, & s = 0. \end{cases} \tag{6.37}$$

Proof. We fix the indices j, k and use the shorthand notation $\Delta f = (\Delta \boldsymbol{f}_j)_k$, $n = n_{j,k}$, $w = w_{j,k}$, $\varsigma = \varsigma_{j,k}$ and similarly with a tilde added for the perturbed sequences. Also set $\boldsymbol{e}_j = \boldsymbol{f}_j - \tilde{\boldsymbol{f}}_j$. For odd points we get

$$e_{j+1,2k+1} = (S\boldsymbol{e}_j)_{2k+1} + \varsigma w(n - \tilde{n}) + \varsigma(w - \tilde{w})\tilde{n}. \tag{6.38}$$

We can estimate

$$\begin{aligned} |n - \tilde{n}|_2 &= \left|\frac{\Delta f}{|\Delta f|_2} - \frac{\Delta \tilde{f}}{|\Delta \tilde{f}|_2}\right|_2 = \left|\frac{\Delta f|\Delta \tilde{f}|_2 - \Delta \tilde{f}|\Delta \tilde{f}|_2 + \Delta \tilde{f}|\Delta \tilde{f}|_2 - \Delta \tilde{f}|\Delta f|_2}{|\Delta \tilde{f}|_2 \; |\Delta f|_2}\right|_2 \\ &\le \frac{|\Delta f - \Delta \tilde{f}|_2 + \left||\Delta \tilde{f}|_2 - |\Delta f|_2\right|}{|\Delta f|_2} \le \frac{2|\Delta f - \Delta \tilde{f}|_2}{|\Delta f|_2} \le \frac{4\,|\boldsymbol{e}_j|_{2,\infty}}{|(\Delta \boldsymbol{x}_j)_k|}. \end{aligned} \tag{6.39}$$

Before continuing, we derive a basic estimate for $|\boldsymbol{e}_{j+1} - S\boldsymbol{e}_j|_\infty$. Set $r = \min(\beta - 1, 1)$. Like in (6.35),

$$|w| \le c|(S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j))_{2k+1}|,$$

and clearly $|\tilde{n}|_2 = 1$. Together with (6.36, 6.38, 6.39) and Lemma 4.6 we then get

$$\begin{aligned} |e_{j+1,2k+1} - (S\boldsymbol{e}_j)_{2k+1}|_2 &\le c|(S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j))_{2k+1}|\frac{|\boldsymbol{e}_j|_{2,\infty}}{|(\Delta \boldsymbol{x}_j)_k|} + |\tilde{w} - w| \\ &\le c\; |\boldsymbol{e}_j|_{2,\infty} \max_{\ell \in I_{2k+1}} \frac{|(\Delta \boldsymbol{x}_j)_\ell|^{1+r}}{|(\Delta \boldsymbol{x}_j)_k|} + |\tilde{w} - w|_\infty \\ &\le c\,|\boldsymbol{e}_j|_{2,\infty} \mathcal{N}(\boldsymbol{x}_j)^B\, |\Delta \boldsymbol{x}_j|_\infty^r + E_w 2^{-js} \\ &\le c\; 2^{-jr} \left|\boldsymbol{x}_j^{[1]}\right|_\infty^r |\boldsymbol{e}_j|_{2,\infty} + E_w 2^{-js}. \end{aligned}$$

Since $f_{j+1,2k} = f_{j,k}$ and $\tilde{f}_{j+1,2k} = \tilde{f}_{j,k}$, we furthermore have $e_{j+1,2k} = e_{j,k}$. From Lemma 6.1 and (6.36) it then follows that

$$|\boldsymbol{e}_{j+1} - S\boldsymbol{e}_j|_{2,\infty} \le c_1\; 2^{-jr}\, |\boldsymbol{e}_j|_{2,\infty} + E_w 2^{-js}, \qquad \forall j \ge 0. \tag{6.40}$$

The proof will now continue through three steps. In the first step we show that the error $\boldsymbol{e}_j$ can grow at most exponentially with j. In the second step we prove that this in fact implies that $\boldsymbol{e}_j$ grows at most polynomially, and finally, in the last step, we show that the polynomial growth actually implies (6.37).

Step 1: Exponential growth of $|\boldsymbol{e}_j|_{2,\infty}$. From (6.40) we have

$$|\boldsymbol{e}_{j+1}|_{2,\infty} \leq |S\boldsymbol{e}_j|_{2,\infty} + c_1\, 2^{-jr}\, |\boldsymbol{e}_j|_{2,\infty} + E_w 2^{-js} \leq (|S|_\infty + c_1)\, |\boldsymbol{e}_j|_{2,\infty} + E_w =: d\, |\boldsymbol{e}_j|_{2,\infty} + E_w,$$

and we assume without loss of generality that $d \geq 2$. By induction, for all j,

$$|\boldsymbol{e}_{j+1}|_{2,\infty} \leq d^{j+1} E_f + E_w \sum_{k=0}^{j} d^k = d^{j+1} E_f + E_w \mathcal{G}\,(d, j+1, 0) \leq c_0\, d^{j+1}(E_f + E_w). \tag{6.41}$$

This shows that $\boldsymbol{e}_j$ cannot grow faster than exponentially. It is not a sharp estimate, and we will use it only as a stepping stone to show the more precise estimate (6.37).

Step 2: Polynomial growth of $|\boldsymbol{e}_j|_{2,\infty}$. We now proceed to show that the error can in fact only grow polynomially with j. As an intermediate result, we will show that $\boldsymbol{e}_j$ is approximately produced by S from $\boldsymbol{e}_0$: the sequences $\boldsymbol{e}_j$ satisfy an estimate of the type (4.1), at least for $j \leq J$ with J finite. We can then apply Theorem 4.2 and bound the growth of the sequences $\boldsymbol{e}_j$ for $j \leq J$.

Let $\nu = \min(r, s)/2 < 1$. By (6.40) we get for $0 \leq j \leq J$,

$$\begin{aligned} |\boldsymbol{e}_{j+1} - S\boldsymbol{e}_j|_{2,\infty} &\leq 2^{-\nu j}\left[c_1\, 2^{-j(r-\nu)}\, |\boldsymbol{e}_k|_{2,\infty} + E_w 2^{-j(s-\nu)}\right] \\ &\leq 2^{-\nu j} \max_{0\leq k\leq J}\left[c_1\, 2^{-k(r-\nu)}\, |\boldsymbol{e}_k|_{2,\infty} + E_w 2^{-k(s-\nu)}\right] =: a_J 2^{-\nu j}. \end{aligned} \tag{6.42}$$

Consider now another family of sequences, $\tilde{\boldsymbol{e}}_j^J$, such that $\tilde{\boldsymbol{e}}_j^J = \boldsymbol{e}_j$ for $0 \leq j \leq J$, and $\tilde{\boldsymbol{e}}_j^J = S\tilde{\boldsymbol{e}}_{j-1}^J$ for $j > J$. Then, by construction,

$$\left|\tilde{\boldsymbol{e}}_{j+1}^{J+1} - S\tilde{\boldsymbol{e}}_j^{J+1}\right|_{2,\infty} \leq a_J 2^{-\nu j}, \qquad \forall j \geq 0,$$

and we can apply Theorem 4.2 to both elements of $\tilde{\boldsymbol{e}}_j^{J+1} \in X^2$ with $a = a_J$, $q = 0$ and ν as above. Since we can choose a p-suitable μ for which $\mu \leq p - 1 < p - \nu$, we have $\varrho = p - \nu \leq p$. Moreover, $\nu = 0$ only when $s = 0$. There is hence a constant $c_2 \geq 1$, independent of E_f and a_J, such that

$$\left|\tilde{\boldsymbol{e}}_j^{J+1}\right|_{2,\infty} \leq c_2\left(\left|\tilde{\boldsymbol{e}}_0^{J+1}\right|_\infty + a_J\right) j^\eta \leq c_2\,(E_f + a_J)\, j^\eta, \qquad \forall j, J > 0, \tag{6.43}$$

with η given as in (6.37). Taking $j = J + 1$ in (6.43) yields

$$|\boldsymbol{e}_j|_{2,\infty} = \left|\tilde{\boldsymbol{e}}_j^j\right|_{2,\infty} \leq c_2\,(E_f + a_{j-1})\, j^\eta, \qquad \forall j \geq 0. \tag{6.44}$$

Let K and K' be two positive integers such that $1 \leq K \leq K'$ and

$$c_1 c_2 2^{-K(r-\nu)} \leq 1, \qquad k^\eta 2^{-(k-K)(r-\nu)} \leq 1, \qquad k \geq K'. \tag{6.45}$$

Then, for $j \geq K'$, using (6.41, 6.45),

$$\begin{aligned} a_j \leq\ & \max\Bigg(\max_{0\leq k\leq K'-1} c_1\, 2^{-k(r-\nu)}\, |\boldsymbol{e}_k|_{2,\infty} + E_w 2^{-k(s-\nu)}, \\ & \frac{1}{c_2}\max_{K'\leq k\leq j} c_1 c_2 2^{-K(r-\nu)} 2^{-(k-K)(r-\nu)}\, |\boldsymbol{e}_k|_{2,\infty} + E_w 2^{-k(s-\nu)}\Bigg) \\ \leq\ & \max\left(\max_{0\leq k\leq K'-1} c_0 c_1\, \left(2^{-(r-\nu)}d\right)^k (E_f + E_w),\ \frac{1}{c_2}\max_{K'\leq k\leq j} 2^{-(k-K)(r-\nu)}\, |\boldsymbol{e}_k|_{2,\infty}\right) + E_w. \end{aligned}$$

But, since $c_2 \geq 1$ and $d \geq 2$,

$$\max_{0\leq k\leq K'-1} c_0 c_1\, \left(2^{-(r-\nu)}d\right)^k = c_0 c_1\, \left(2^{-(r-\nu)}d\right)^{K'-1} < c_0 c_1 c_2\, \left(2^{-(r-\nu)}d\right)^{K'} \leq c_1 c_2\, 2^{-K(r-\nu)} c_0 d^{K'},$$

and by (6.45),

$$a_j \leq \max\left(c_0 d^{K'}(E_f+E_w),\ \frac{1}{c_2}\max_{K'\leq k\leq j} 2^{-(k-K)(r-\nu)}\left|\boldsymbol{e}_k\right|_{2,\infty}\right)+E_w, \tag{6.46}$$

$$\leq \max\left(c_0 d^{K'}(E_f+E_w),\ \max_{K'\leq k\leq j}\frac{\left|\boldsymbol{e}_k\right|_{2,\infty}}{c_2 k^{\eta}}\right)+E_w, \tag{6.47}$$

for $j\geq K'$. We claim that (6.41, 6.44, 6.47) yields the polynomial bound on the growth of $\boldsymbol{e}_j$ given by

$$\left|\boldsymbol{e}_j\right|_{2,\infty}\leq c_2(E_w+E_f)j^{\eta}\Big[j-K'+c_0 d^{K'}\Big], \tag{6.48}$$

for $j\geq K'$. This is clearly true for $j=K'$ by (6.41) and $c_2\geq 1$, $K'\geq 1$. Suppose (6.48) holds for j with $K'\leq j\leq n$. We then have, from (6.44, 6.47, 6.48) and (6.45),

$$\begin{aligned}
\left|\boldsymbol{e}_{n+1}\right|_{2,\infty} &\leq c_2\left(E_f+a_n\right)(n+1)^{\eta}\leq c_2\left(E_f+E_w\right)(n+1)^{\eta}\\
&\quad+\max\left(c_2 c_0 d^{K'}(E_f+E_w),\ \max_{K'\leq k\leq n}\frac{\left|\boldsymbol{e}_k\right|_{2,\infty}}{k^{\eta}}\right)(n+1)^{\eta}\\
&\leq c_2\left(E_f+E_w\right)(n+1)^{\eta}\\
&\quad+c_2(E_f+E_w)(n+1)^{\eta}\max\left(c_0 d^{K'},\ \max_{K'\leq k\leq n}\Big[k-K'+c_0 d^{K'}\Big]\right)\\
&= c_2(E_w+E_f)(n+1)^{\eta}\Big[n+1-K'+c_0 d^{K'}\Big].
\end{aligned}$$

This shows (6.48) for all $j\geq K'$.

Step 3: Conclusion. It is now rather easy to see that (6.48) implies (6.37). We note first that there is a constant c_3 such that

$$2^{-(k-K)(r-\nu)}k^{\eta}(k-K')\leq c_3,\qquad \forall k\geq K'.$$

It then follows from (6.45, 6.46, 6.48) that for $j\geq K'$

$$\begin{aligned}
a_j &\leq \max\left(c_0 d^{K'}(E_f+E_w),\ \max_{K'\leq k\leq j}2^{-(k-K)(r-\nu)}(E_w+E_f)k^{\eta}\Big[k-K'+c_0 d^{K'}\Big]\right)+E_w\\
&\leq (E_f+E_w)\max\left(c_0 d^{K'},\ c_3+c_0 d^{K'}\right)+E_w\leq C(E_f+E_w).
\end{aligned}$$

On the other hand, for $0\leq j<K'$, we get from (6.42) and (6.41),

$$\begin{aligned}
a_j &\leq \max_{0\leq k\leq K'}\Big[c_0 c_1\, 2^{-k(r-\nu)}d^k(E_f+E_w)+E_w 2^{-k(s-\nu)}\Big]\\
&\leq c_0 c_1\, 2^{-K'(r-\nu)}d^{K'}(E_f+E_w)+E_w\leq C(E_f+E_w).
\end{aligned}$$

The stability (6.37) then follows from (6.44). □

Remarks:

1. The result in Theorem 6.4 shows that setting wavelet coefficients $w_{j,k}$ to zero beyond a certain level, leads to a stable compression scheme. For example, suppose that the wavelet coefficients decay as

$$\left|\boldsymbol{w}_j\right|_{\infty}\leq c\, 2^{-j\lambda},\qquad \lambda>0, \tag{6.49}$$

   and that we set

$$\tilde{\boldsymbol{w}}_j=\begin{cases}\boldsymbol{w}_j, & 0\leq j\leq J,\\ 0, & j>J.\end{cases}$$

as well as $\tilde{\boldsymbol{f}}_0 = \boldsymbol{f}_0$. This corresponds to doing the reconstruction of the curve exactly in the first J levels, and to use only the pure subdivision scheme for further refinement of the curve. Theorem 6.4 can then be applied at level J with $E_f = 0$, $E_w = c\ 2^{-J\lambda}$ and $s = \lambda$, so that

$$\left|\boldsymbol{f}_j - \tilde{\boldsymbol{f}}_j\right|_{2,\infty} \leq C\ 2^{-J\lambda}, \qquad \forall j \geq 0. \tag{6.50}$$

If we are in the setting described in Theorems 5.7 and 5.8 we know from (5.30) that the non-uniformity is controlled at level J, hence $\mathcal{N}(\boldsymbol{x}_J) < R$ where R is the bound for the weakly contractive scheme S. ($R = \infty$ for $S = S_2$). When we proceed beyond J using only S for refinement, $\tilde{\boldsymbol{x}}_j$ remains strictly increasing and $\mathcal{N}(\tilde{\boldsymbol{x}}_j) < R$ trivially, since S is weakly contractive. In particular, we can define the perturbed function $\tilde{\gamma}_j^J(x)$ as the piecewise linear interpolant of $\tilde{\boldsymbol{f}}_j = (\tilde{\boldsymbol{x}}_j,\ \tilde{\boldsymbol{y}}_j)$ for all j and denote by $\tilde{\gamma}^J(x)$ the limiting function obtained when $j \to \infty$. It then follows from (6.50) that

$$\left|\gamma - \tilde{\gamma}^J\right|_\infty \leq c\ 2^{-J\lambda}.$$

2. One can also consider a thresholding scheme. Let us again assume (6.49) and set

$$\tilde{w}_{j,k} = \begin{cases} w_{j,k}, & |w_{j,k}| \geq \varepsilon, \\ 0, & |w_{j,k}| < \varepsilon. \end{cases}$$

We then have $|w_{j,k} - \tilde{w}_{j,k}| \leq \varepsilon$ for all j,k, as well as $|w_{j,k} - \tilde{w}_{j,k}| \leq |w_{j,k}| \leq c2^{-j\lambda}$. It follows that, for $0 \leq \kappa \leq 1$,

$$|w_{j,k} - \tilde{w}_{j,k}| \leq \varepsilon^{1-\kappa} c^\kappa 2^{-j\lambda\kappa}.$$

If $\boldsymbol{f}_0 = \tilde{\boldsymbol{f}}_0$ and $\kappa > 0$, then we obtain from Theorem 6.4 that

$$\left|\boldsymbol{f}_j - \tilde{\boldsymbol{f}}_j\right|_{2,\infty} \leq C'\ c^\kappa \epsilon^{1-\kappa}, \qquad \kappa > 0, \tag{6.51}$$

where C' depends on the product $\lambda\kappa$.

In this setting we cannot be sure that $\tilde{\boldsymbol{x}}_j$ remains strictly increasing for all j, and we cannot define the functions $\tilde{\gamma}_j(x)$ as above. The normal parameterization of the perturbed curve remains well-defined however, and we let $\tilde{\mathrm{x}}_j(t)$ and $\tilde{\mathrm{y}}_j(t)$ be the piecewise linear interpolants of $(2^{-j}\boldsymbol{k},\ \tilde{\boldsymbol{x}}_j)$ and $(2^{-j}\boldsymbol{k},\ \tilde{\boldsymbol{y}}_j)$ respectively. We denote the limits as $j \to \infty$ by $\tilde{\mathrm{x}}(t)$ and $\tilde{\mathrm{y}}(t)$, and also set $\tilde{\Gamma}(t) := (\tilde{\mathrm{x}}(t),\ \tilde{\mathrm{y}}(t))$. Then (6.51) shows that,

$$\sup_t |\Gamma(t) - \tilde{\Gamma}(t)|_2 = \sup_t \left||\mathrm{x}(t) - \tilde{\mathrm{x}}(t)|^2 + |\mathrm{y}(t) - \tilde{\mathrm{y}}(t)|^2\right|^{1/2} \leq C'\ c^\kappa \epsilon^{1-\kappa}.$$

7. Examples

We conclude this paper with several examples. We first look at the class of Lagrangian interpolating subdivision predictors S_{2l}. The predictor S_{2l} uses order $2l-1$ Lagrange interpolation. The simplest case, $l = 1$, corresponds to the midpoint rule; for $l = 2$ one obtains the well-known 4-point scheme; we also consider $l = 3$ and $l = 4$ here. The width of the scheme increases with l ($B = 4l-2$), as well as the regularity of the limit function. This therefore provides a nice test family to check the dependence on σ of the decay of the normal wavelet coefficients or the smoothness of the parameterization. We also consider on hybrid case from [7]. Finally we study an example to test some of the restrictions imposed in the theorems. In particular, we often require that the function γ be C^β with $\beta > 1$. We introduce a curve in Section 7.2 that is merely Lipschitz ($\beta = 1$), for which many of the conclusions of our theorems do not hold, showing that $\beta > 1$ is necessary.

7.1. The Lagrange interpolation prediction schemes $S_{2\ell}$.

7.1.1. *Two point scheme.* In the linear case we simply have

$$\left|S_2^{[1]}\boldsymbol{x}_j^{[1]}\right|_\infty = \left|\boldsymbol{x}_j^{[1]}\right|_\infty.$$

Hence $\mu = 0$ is p-suitable for $p = 1$, so $\hat\sigma \geq 1$. If $\gamma \in C^2$ we have $Q \geq 1$ in Theorem 6.3. The worst case is $Q = 1$, for which $P = 0$, $\kappa = 1$ and $\eta = 0$. Then $\mathtt{x}(t) \in \mathrm{Lip}^1$, which is optimal. Theorem 6.3 also predicts that the wavelet coefficients decay as $\mathrm{O}(2^{-2j})$.

7.1.2. *Four point scheme.* For the cubic case we start from the estimate

$$\left|S_4^{[3]}\boldsymbol{x}_j^{[3]}\right|_\infty \leq 2\left|\boldsymbol{x}_j^{[3]}\right|_\infty.$$

Hence $\mu = 1$ is p-suitable for $p = 3$, so $\hat\sigma \geq 2$. If $\gamma \in C^2$, Theorem 6.3 gives $Q = 2$ and $\mathtt{x}(t) \in C^{2^-}$, which again is optimal. If $\gamma \in C^{3+\varepsilon}$ we get $Q \geq 2$ and Theorem 6.3 predicts that the wavelet coefficients decay as $\mathrm{O}(j2^{-3j})$.

7.1.3. *The 2-4 hybrid scheme.* Let

$$S_w = (1-w)S_2 + wS_4, \qquad 0 < w \leq 1.$$

This convex combination of S_2 and S_4 is of order two and $\left|S_w^{[2]}\right|_\infty = 2$, hence $\hat\sigma \geq 2 - \log_2 2 = 1$, which is not sufficient to prove that $\mathtt{x}(t) \in C^1$. However, we can also use the fact that

$$\left|\left(S_w^{[2]}\right)^2\right|_\infty = 4 - w.$$

This means that $\hat\sigma \geq 2 - \frac{1}{2}\log_2(4-w) > 1$ for $w > 0$, so $\mathtt{x}(t) \in C^1$ by Theorem 6.3 when $\gamma \in C^2$.

7.1.4. *Six point scheme.* For the six point scheme we start from the estimate

$$\left|S_6^{[5]}\boldsymbol{x}_j^{[5]}\right|_\infty \leq 4.75\left|\boldsymbol{x}_j^{[5]}\right|_\infty.$$

Hence $\mu = \log_2 4.75$ is p-suitable for $p = 5$, so $\hat\sigma \geq 5 - \log_2 4.75 \approx 2.75$. If $\gamma \in C^3$, Theorem 6.3 gives $Q \geq 5 - \log_2 4.75 \approx 2.75$ and $\mathtt{x}(t) \in C^{2.75}$, which is not optimal. If $\gamma \in C^4$, Theorem 6.3 predicts that the wavelet coefficients decay as $\mathrm{O}(2^{-3.75j})$.

One can show that the six point scheme actually has a limit function with regularity 2.83. (This value is obtained by L^1-estimates of the decay of the Fourier transform of the limit function. Because the mask of the subdivision scheme defines a nonnegative trigonometric polynomial for Lagrange interpolating subdivision, the Fourier transform of the limit function is positive as well, so that this L^1-estimate can be shown to be optimal.)

7.1.5. *Eight point scheme.* For the eight point scheme we start from the estimate

$$\left|S_8^{[7]}\boldsymbol{x}_j^{[7]}\right|_\infty \leq 13\left|\boldsymbol{x}_j^{[7]}\right|_\infty.$$

Hence $\mu = \log_2 13$ is p-suitable for $p = 7$, so $\hat\sigma \geq 7 - \log_2 13 \approx 3.30$. If $\gamma \in C^4$, Theorem 6.3 gives $Q \geq 7 - \log_2 13 \approx 3.30$ and $\mathtt{x}(t) \in C^{3.30}$, which again is not optimal. If $\gamma \in C^5$, Theorem 6.3 predicts that the wavelet coefficients decay as $\mathrm{O}(2^{-4.30j})$. Using Fourier methods one can show that the optimal regularity of the limit function is 3.55.

Numerical examples for S_2 till S_8 are given in Figure 7. It shows that the observed behavior is very close to the precise theoretical prediction.

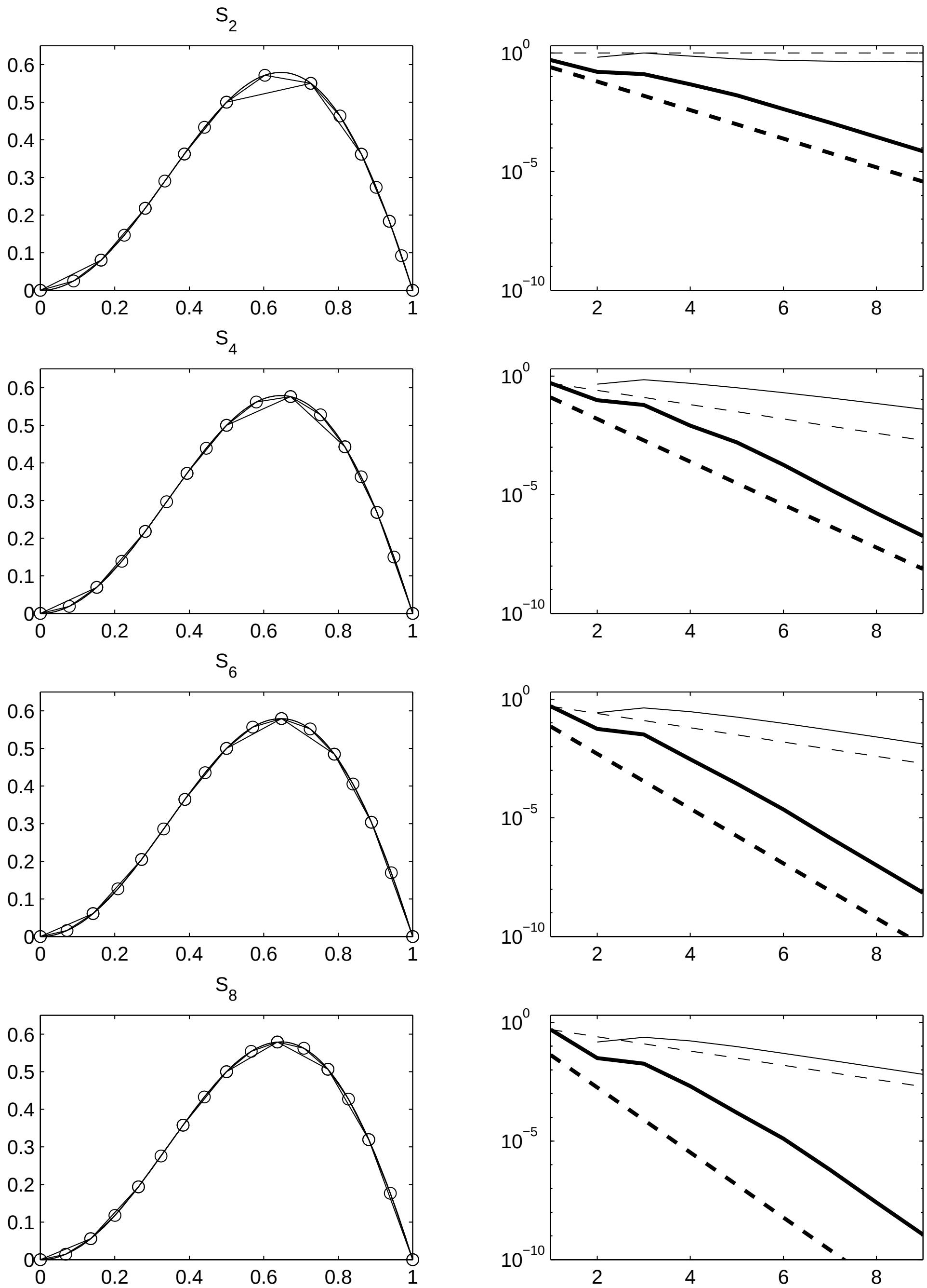

FIGURE 7. Numerical study of the first Lagrange interpolation subdivision schemes, S_2 to S_8. Left column shows the normal multiresolution approximation at levels $j = 3, 4$. Right column shows the decay of wavelet coefficients as a function of level j (solid, bold) compared with the function $2^{-j(Q+1)}$ with $Q = 1, 2, 2.83, 3.55$ (dashed, bold) together with the non-uniformity measure $\mathcal{N}(\boldsymbol{x}_j)$ as a function of j (solid) compared with the function $2^{-j\min(Q-1,1)}$ (dashed). In these figures it is important to compare the *slopes* of the solid and dashed lines.

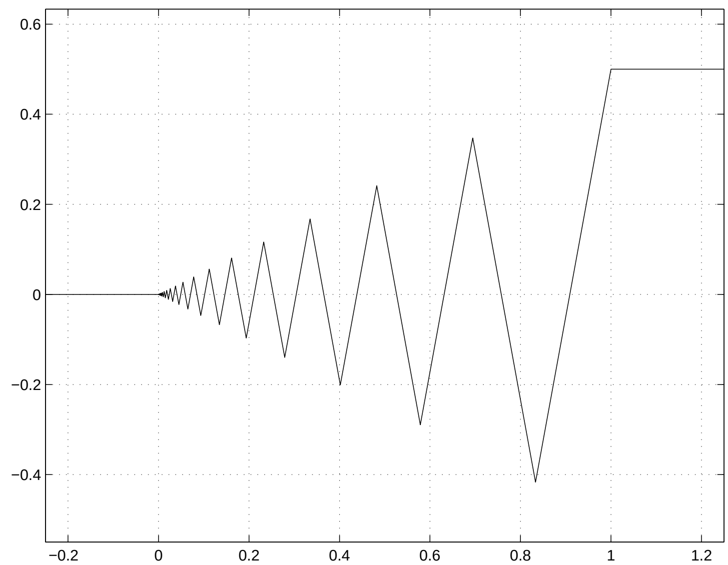

FIGURE 8. The curve in Section 7.2 for $r = 1/2$.

7.2. **A counterexample if $\beta \le 1$.** We will here show an example of a Lipschitz curve for which results in the earlier sections break down. This indicates that the requirement in the theorems that $\gamma \in C^\beta$ with $\beta > 1$ is close to optimal.

Take r such that $0 \le r < \frac{1}{\sqrt{3}}$ and set

$$\alpha = \frac{1 + r^2}{2(1 - r^2)}.$$

It is easy to see that α is an increasing function of r and that $0 \le r < 1/\sqrt{3}$ implies $1/2 \le \alpha < 1$. The curve we consider is inspired by the graph of an increasingly oscillating function, such as

$$\varphi(x) = rx\cos(\pi \log x / \log \alpha) \tag{7.1}$$

for $0 < x < 1$ and constant for other x. For simplicity we consider a piecewise linear approximation of φ, for which one gets simple formulae for all the quantities in which we are interested. More precisely, for $0 < x \le 1$, let $\gamma(x)$ be the piecewise linear interpolant of the points (t_ℓ, u_ℓ) given by

$$t_\ell = \alpha^\ell, \qquad u_\ell = rt_\ell \cos(\pi \log t_\ell / \log \alpha) = rt_\ell(-1)^\ell, \qquad \ell \ge 0.$$

For $x \le 0$, set $\gamma(x) \equiv 0$ and $\gamma(x) \equiv r$ for $x > 1$. The curve is illustrated in Figure 8 for $r = 1/2$.

We note that γ is a continuous bounded curve, well defined when r is in the stated interval. Moreover, recalling that $\alpha < 1$, we see that γ is both Lipschitz continuous,

$$\Omega(1, \gamma) = \sup_{x,x'} \left| \frac{\gamma(x) - \gamma(x')}{x - x'} \right| = \sup_\ell \left| \frac{u_{\ell+1} - u_\ell}{t_{\ell+1} - t_\ell} \right| = \sup_\ell \; r \left| \frac{t_{\ell+1} + t_\ell}{t_{\ell+1} - t_\ell} \right| = r\frac{1 + \alpha}{1 - \alpha},$$

and of bounded variation,

$$\mathrm{TV}(\gamma) = \sum_{\ell=0}^{\infty} |u_{\ell+1} - u_\ell| = r\sum_{\ell=0}^{\infty} |t_{\ell+1} + t_\ell| = r(1 + \alpha)\sum_{\ell=0}^{\infty} \alpha^\ell = r\frac{1 + \alpha}{1 - \alpha}.$$

It is clearly not C^1, though. (Neither is φ.)

We apply the normal scheme to the curve, with $\boldsymbol{x}_0 = \boldsymbol{k}$ and the two point subdivision scheme S_2 as predictor. To go from level j to level $j+1$ we need to solve the equation in (5.1),

$$\left(x_{j+1,2k+1} - \frac{x_{j,k} + x_{j,k+1}}{2}\right)(x_{j,k+1} - x_{j,k}) + \left(y_{j+1,2k+1} - \frac{y_{j,k} + y_{j,k+1}}{2}\right)(y_{j,k+1} - y_{j,k}) = 0. \tag{7.2}$$

There are in general many solutions to this equation for which $\Delta \boldsymbol{x}_{j+1} > 0$ and we are free to set a rule telling which one to pick. In particular we can take the solution that is furthest away from the predicted point. In that case we will have $x_{j,0} = 0$ and $x_{j,1} = t_j$ for all j. This follows by induction after inserting these expressions into (7.2) with $k = 0$,

$$\begin{aligned} \left(t_{j+1} - \frac{t_j}{2}\right) t_j + \left(u_{j+1} - \frac{u_j}{2}\right) u_j &= \left(\alpha - \frac{1}{2}\right)\alpha^{2j} + \left(-\alpha - \frac{1}{2}\right) r^2 \alpha^{2j} \\ &= \alpha^{2j}\left(\alpha(1 - r^2) - \frac{1}{2}(1 + r^2)\right) = 0. \end{aligned}$$

Since γ is linear between the interpolation points (t_ℓ) it also follows by induction that the j-th level normal multiresolution approximation $\gamma_j(t)$ is *exact* for $t \le 0$ and $t \ge t_j$. This means that the wavelet coefficients are zero for $k \neq 0$. For $k = 0$,

$$(t_{j+1}, u_{j+1}) - \frac{1}{2}(t_j, u_j) = \alpha^j \left(\alpha - \frac{1}{2},\ r(-1)^{j+1}\left(\alpha + \frac{1}{2}\right)\right) = \frac{\alpha^j r}{1 - r^2}\left(r,\ (-1)^{j+1}\right). \tag{7.3}$$

Hence,

$$|\boldsymbol{w}_j|_\infty = |w_{j,0}| = \frac{r\sqrt{1 + r^2}}{1 - r^2}\alpha^j,$$

and since we can pick α as close to one as we like, we can indeed construct Lipschitz continuous (and BV) curves where the normal scheme has arbitrarily slow exponential convergence. Similar estimates would hold if we chose γ to be the graph of φ defined by (7.1).

This curve also provides a counter example to a few other results in the earlier sections of this paper. We note that

$$(\Delta \boldsymbol{x}_j)_k = \begin{cases} 2^{-j}, & k < 0 \text{ or } k \ge 2^j, \\ t_j, & k = 0, \\ 2^{-i}(t_{j-i-1} - t_{j-i}), & 2^i \le k < 2^{i+1},\ 0 \le i < j, \end{cases} = \begin{cases} 2^{-j}, & k < 0 \text{ or } k \ge 2^j, \\ \alpha^j, & k = 0, \\ \alpha^j \frac{1-\alpha}{2^i \alpha^{i+1}}, & 2^i \le k < 2^{i+1},\ 0 \le i < j, \end{cases} \tag{7.4}$$

and since $\alpha \ge 1/2$,

$$\alpha^j \frac{1 - \alpha}{2^i \alpha^{i+1}} = \alpha^j \left(\frac{1}{\alpha} - 1\right)(2\alpha)^{-i} \le \alpha^j \ge 2^{-j},$$

so

$$|\Delta \boldsymbol{x}_j|_\infty = (\Delta \boldsymbol{x}_j)_0 = \alpha^j, \qquad \left|\boldsymbol{x}_j^{[1]}\right|_\infty = (2\alpha)^j.$$

By taking $\alpha > 1/2$, this shows that Lemma 6.1 does not hold when γ is just Lipschitz.

Furthermore, just like for the wavelet coefficients $(S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j))_k = 0$ for $k \neq 1$. Suppose we let

$$\alpha = 2^{-1/2n}, \qquad n \in \mathbb{Z}^+,$$

so that

$$\frac{t_j}{2} = 2^{\frac{-j-2n}{2n}} = \alpha^{j+2n} = t_{j+2n}.$$

Then

$$(S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j))_1 = \frac{\gamma(t_j)}{2} - \gamma\left(\frac{t_j}{2}\right) = \frac{\gamma(t_j)}{2} - \gamma\left(t_{j+2n}\right) = r(-1)^j\left(\frac{t_j}{2} - t_{j+2n}(-1)^{2n}\right) = 0.$$

Therefore,

$$|S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty = 0, \qquad \forall j > 0.$$

But, by (7.3)

$$|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty = \alpha^j \frac{r^2}{1-r^2} > 0, \qquad \forall j > 0,$$

so $|\boldsymbol{x}_{j+1} - S\boldsymbol{x}_j|_\infty > |S\gamma(\boldsymbol{x}_j) - \gamma(S\boldsymbol{x}_j)|_\infty$ for all j, eventhough $|\Delta\boldsymbol{x}_j|_\infty \to 0$. Hence, Lemma 5.5 is not true when γ is Lipschitz.

Finally, the last statement of Theorem 5.8 breaks down for Lipschitz curves. By (7.4)

$$\mathcal{N}(\boldsymbol{x}_j) \geq \frac{(\Delta\boldsymbol{x}_j)_0}{(\Delta\boldsymbol{x}_j)_{-1}} = (2\alpha)^j,$$

which blows up if we pick $\alpha > 1/2$.

8. Open Questions

The work in this paper was motivated by the application on normal meshes in surface representation and compression. In [13] it was noted numerically that normal meshes are stable, yield smooth parameterization, and allow for superior compression. In this paper these observations where proven theoretically for curves. More work needs to be done for the case of surfaces. For fairly smooth surfaces, C^1 and beyond, we expect many of the results of this paper to carry through. However, the more interesting question is are how normal meshes work for less smooth spaces, particularly spaces that are used to model natural images such as BV, and the Besov space $B^1_{1,1}$.

So far normal meshes have only involved interpolating subdivision schemes. It is well known that both in the curve and surface case, non-interpolating or approximating, subdivision schemes not only yield smoother functions for a fixed support, but also result in fewer oscillations or more "fair" limit functions. Therefore non-interpolating schemes are preferred in practice. Interesting open problem are the construction of normal multiresolution for non-interpolating subdivision and the use of the approximating subdivision machinery in this paper to study its properties.

After finishing this paper we learned of the work of Maarten Jansen et al. [12]. They use normal meshes to approximate piecewise continuous height fields and observe that normal meshes have the capability to adaptively approximate the jump in a way similar to wedgelets and curvelets. They conjecture that for the class of so-called "Horizon images" normal meshes converge as N^{-1} instead of the regular wavelet rate of $N^{-0.5}$. They show that for piecewise continuous functions, the average L^2 decay is even $N^{-1.26}$ again compared to $N^{-0.5}$ for regular wavelets. This shows that to study normal multiresolution, the class of piecewise continuous functions may be more appropriate than the larger Lipschitz class we considered.

Acknowledgement. We would like to thank Ruud van Damme for communicating an unpublished contractivity result to us. The research for this paper was carried out while Olof Runborg held a postdoctoral appointment with the Program for Applied and Computational Mathematics at Princeton University, supported by NSF KDI grant DMS-9872890. Ingrid Daubechies also gratefully acknowledges partial support by this grant, as well as by NSF AFOSR F49620-01-1-0099, and NSF DMS-0070689.

References

[1] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli. Stationary subdivision. *Memoirs Amer. Math. Soc.*, 93(453), 1991.

[2] I. Daubechies, I. Guskov, and W. Sweldens. Regularity of irregular subdivision. *Constr. Approx.*, 15:381–426, 1999.

[3] I. Daubechies and J. C. Lagarias. Two-scale difference equations I. Existence and global regularity of solutions. *SIAM J. Math. Anal.*, 22(5):1388–1410, 1991.

[4] I. Daubechies and J. C. Lagarias. Two-scale difference equations II. Local regularity, infinite products of matrices and fractals. *SIAM J. Math. Anal.*, 23(4):1031–1079, 1992.

[5] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5(1):49–68, 1989.

[6] N. Dyn, J. Gregory, and D. Levin. Analysis of uniform binary subdivision schemes for curve design. *Constr. Approx.*, 7:127–147, 1991.

[7] N. Dyn, D. Levin, and J. Gregory. A 4-point interpolatory subdivision scheme for curve design. *Comput. Aided Geom. Des.*, 4:257–268, 1987.

[8] A. Finkelstein and D. H. Salesin. Multiresolution curves. In *Computer Graphics (SIGGRAPH '94 Proceedings)*, pages 261–268, 1994.

[9] D. R. Forsey and R. H. Bartels. Hierarchical B-spline refinement. In *Computer Graphics (SIGGRAPH '88 Proceedings)*, pages 205–212, 1988.

[10] S. J. Gortler and M. F. Cohen. Hierarchical and variational geometric modeling with wavelets. In *Proceedings Symposium on Interactive 3D Graphics*. Siggraph, May 1995.

[11] I. Guskov, K. Vidimče, W. Sweldens, and P. Schröder. Normal meshes. In *Computer Graphics (SIGGRAPH '00 Proceedings)*, pages 259–268, 2000.

[12] M. Jansen, R. Baraniuk, and S. Lavu. Multiscale approximation of piecewise smooth functions, using normal offsets. In preparation, 2002.

[13] A. Khodakovsky and I. Guskov. Normal mesh compression. `http://www.multires.caltech.edu/pubs/`, 2000.

[14] A. Khodakovsky, P. Schröder, and W. Sweldens. Progressive geometry compression. In *Computer Graphics (SIGGRAPH '00 Proceedings)*, pages 271–278, 2000.

[15] F. Kuijt and R. van Damme. Monotonicity preserving interpolatory subdivision schemes. *J. Comput. Appl. Math.*, 101(1-2):203–229, 1999.

[16] M. Lounsbery, T. D. DeRose, and J. Warren. Multiresolution surfaces of arbitrary topological type. *ACM Trans. on Graphics*, 16(1):34–73, 1997.

[17] O. Runborg. Weakly contractive subdivision schemes. In preparation, 2002.

[18] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM J. Math. Anal.*, 29(2):511–546, 1997.

[19] R. van Damme. Personal communication. October 2000.

[20] D. Zorin, P. Schröder, and W. Sweldens. Interactive multiresolution mesh editing. In *Computer Graphics (SIGGRAPH '97 Proceedings)*, pages 259–268, 1997.

Appendix A. Proof of Proposition 4.1

For $a > 1$, we have

$$\frac{\mathcal{G}(a,n,\alpha)}{n^\alpha a^n} = \sum_{k=0}^{n-1} \left(\frac{k}{n}\right)^\alpha a^{k-n} = \sum_{k=0}^{n-1} \left(\frac{n-k}{n}\right)^\alpha a^{-k} \le \sum_{k=0}^{n-1} a^{-k} \le \sum_{k=0}^{\infty} a^{-k} = \frac{a}{a-1}.$$

For $a = 1$,

$$\frac{\mathcal{G}(a,n,\alpha)}{n^{\alpha+1}} = \frac{1}{n}\sum_{k=0}^{n-1}\left(\frac{k}{n}\right)^\alpha \le \frac{1}{n}\sum_{k=0}^{n-1} 1 = 1.$$

And for $a < 1$,

$$\mathcal{G}(a,n,\alpha) \le \left(\sup_{k\ge 0} k^\alpha a^{k/2}\right)\sum_{k=0}^{n-1} a^{k/2} \le \frac{\sup_{k\ge 0} k^\alpha a^{k/2}}{1-\sqrt{a}} \le C(a,\alpha).$$

The fact that $\mathcal{G}$ is increasing is obvious from its definition. If $a < b$ there is an N such that when $k > N$ we have $k^{\alpha_1-\alpha_2}(a/b)^k < 1$ and consequently

$$\mathcal{G}(a,n,\alpha_1) = \sum_{k=0}^{n-1} k^{\alpha_1-\alpha_2}\left(\frac{a}{b}\right)^k k^{\alpha_2} b^k \le \mathcal{G}(b,n,\alpha_2)\max_{k\le N} k^{\alpha_1-\alpha_2}\left(\frac{a}{b}\right)^k$$

which shows (4.5). For the last inequality, we have

$$\sum_{k=n_1}^{n_2-1} k^\alpha a^k = n_1^\alpha a^{n_1} + (1+n_1)^\alpha a^{n_1}\sum_{k=1}^{n_2-n_1-1}\left(\frac{k+n_1}{1+n_1}\right)^\alpha a^k \le n_1^\alpha a^{n_1} + 2(1+n_1)^\alpha a^{n_1}\mathcal{G}(a, n_2-n_1, \alpha),$$

showing (4.6).

Appendix B. Weak contractivity of S_4

It is clear that S_2 is weakly contractive with bound $R = \infty$. In this appendix we show that any convex combination of S_2 and S_4 is also weakly contractive with a bound in the range $R \in [3+2\sqrt{2}, \infty)$. This result, as well as an outline of the proof, was communicated to us by Ruud van Damme [19].

Proposition B.1. *The subdivision scheme*

$$S_w = (1-w)S_2 + wS_4, \qquad 0 < w \le 1,$$

is weakly contractive with bound

$$R = \frac{4}{w}\left(1+\sqrt{1-\frac{w}{2}}\right) - 1. \tag{B.1}$$

Proof. As in Section 5 let $\boldsymbol{x}$ denote the initial level and $\tilde{\boldsymbol{x}}$ the level after one refinement, so that $\tilde{\boldsymbol{x}} = S_w\boldsymbol{x}$. Moreover, let $\nu = \mathcal{N}(\boldsymbol{x})$. Finally, we set

$$v_\ell := \frac{w}{8}\left((\Delta\boldsymbol{x})_{\ell+1} - (\Delta\boldsymbol{x})_{\ell-1}\right).$$

We then need to show that if $\nu \le R$, then $\tilde{\boldsymbol{x}}$ is strictly increasing and

$$\max\left(\frac{(\Delta\tilde{\boldsymbol{x}})_k}{(\Delta\tilde{\boldsymbol{x}})_{k+1}}, \frac{(\Delta\tilde{\boldsymbol{x}})_{k+1}}{(\Delta\tilde{\boldsymbol{x}})_k}\right) \le \nu, \qquad \forall k. \tag{B.2}$$

To show that $\tilde{\boldsymbol{x}}$ is strictly increasing, we start by estimating v_ℓ in terms of $(\Delta\boldsymbol{x})_\ell$,

$$\begin{aligned} |v_\ell| &= (\Delta\boldsymbol{x})_\ell \frac{w}{8}\left|\frac{(\Delta\boldsymbol{x})_{\ell-1}}{(\Delta\boldsymbol{x})_\ell} - \frac{(\Delta\boldsymbol{x})_{\ell+1}}{(\Delta\boldsymbol{x})_\ell}\right| \le (\Delta\boldsymbol{x})_\ell\frac{w}{8}\left(\nu - \frac{1}{\nu}\right) \\ &= (\Delta\boldsymbol{x})_\ell\frac{w}{8}\frac{(\nu-1)(\nu+1)^2}{(\nu+1)\nu} \le (\Delta\boldsymbol{x})_\ell\frac{\nu-1}{\nu+1}\frac{(R+1)^2}{R}\frac{w}{8}. \end{aligned}$$

The last step follows since $(x+1)^2/x$ is increasing on $[1,\infty)$. We note moreover that

$$(R+1)^2 = \frac{16}{w^2}\left(1+\sqrt{1-\frac{w}{2}}\right)^2 = \frac{16}{w^2}\left(2-\frac{w}{2}+2\sqrt{1-\frac{w}{2}}\right) = \frac{8}{w}R, \tag{B.3}$$

so

$$|v_\ell| \le (\Delta \boldsymbol{x})_\ell \frac{\nu-1}{\nu+1}. \tag{B.4}$$

Now, let $\ell = \lfloor k/2 \rfloor$. We have from (B.4)

$$\begin{aligned} 2(\Delta\tilde{\boldsymbol{x}})_k &= (S_w^{[1]}\Delta\boldsymbol{x})_k = (\Delta\boldsymbol{x})_\ell + \frac{w}{8}\begin{cases}(\Delta\boldsymbol{x})_{\ell-1} - (\Delta\boldsymbol{x})_{\ell+1}, & k=2\ell,\\ -(\Delta\boldsymbol{x})_{\ell-1} + (\Delta\boldsymbol{x})_{\ell+1}, & k=2\ell+1,\end{cases} \\ &\ge (\Delta\boldsymbol{x})_\ell - |v_\ell| \ge (\Delta\boldsymbol{x})_\ell\left(1-\frac{\nu-1}{\nu+1}\right) = \frac{2(\Delta\boldsymbol{x})_\ell}{\nu+1} > 0. \end{aligned}$$

This shows that $\tilde{\boldsymbol{x}}$ is strictly increasing.

Suppose now that $k = 2\ell$. Then, again by (B.4),

$$\begin{aligned} \max\left(\frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}, \frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell}}\right) &= \max\left(\frac{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell}}{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+1}}, \frac{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+1}}{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell}}\right) \\ &= \max\left(\frac{(\Delta\boldsymbol{x})_\ell - v_\ell}{(\Delta\boldsymbol{x})_\ell + v_\ell}, \frac{(\Delta\boldsymbol{x})_\ell + v_\ell}{(\Delta\boldsymbol{x})_\ell - v_\ell}\right) \\ &= \frac{(\Delta\boldsymbol{x})_\ell + |v_\ell|}{(\Delta\boldsymbol{x})_\ell - |v_\ell|} \le \frac{1+\frac{\nu-1}{\nu+1}}{1-\frac{\nu-1}{\nu+1}} = \nu, \end{aligned}$$

proving (B.2) for k even. Suppose next that $k = 2\ell+1$. Then

$$\begin{aligned} \max\left(\frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell+2}}, \frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell+2}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}\right) &= \max\left(\frac{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+1}}{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+2}}, \frac{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+2}}{(S_w^{[1]}\Delta\boldsymbol{x})_{2\ell+1}}\right) \\ &= \max\left(\frac{(\Delta\boldsymbol{x})_\ell + v_\ell}{(\Delta\boldsymbol{x})_{\ell+1} - v_{\ell+1}}, \frac{(\Delta\boldsymbol{x})_{\ell+1} - v_{\ell+1}}{(\Delta\boldsymbol{x})_\ell + v_\ell}\right). \end{aligned}$$

Moreover,

$$\begin{aligned} \nu v_\ell + v_{\ell+1} &= \frac{w}{8}(-\nu(\Delta\boldsymbol{x})_{\ell-1} + \nu(\Delta\boldsymbol{x})_{\ell+1} - (\Delta\boldsymbol{x})_\ell + (\Delta\boldsymbol{x})_{\ell+2}) \\ &\ge \frac{w}{8}(-\nu^2(\Delta\boldsymbol{x})_\ell + \nu(\Delta\boldsymbol{x})_{\ell+1} - (\Delta\boldsymbol{x})_\ell + (\Delta\boldsymbol{x})_{\ell+1}/\nu) = -\frac{w}{8}\left(\nu+\frac{1}{\nu}\right)(\nu(\Delta\boldsymbol{x})_\ell - (\Delta\boldsymbol{x})_{\ell+1}), \end{aligned}$$

and

$$\begin{aligned} v_\ell + \nu v_{\ell+1} &= \frac{w}{8}(-(\Delta\boldsymbol{x})_{\ell-1} + (\Delta\boldsymbol{x})_{\ell+1} - \nu(\Delta\boldsymbol{x})_\ell + \nu(\Delta\boldsymbol{x})_{\ell+2}) \\ &\le \frac{w}{8}(-(\Delta\boldsymbol{x})_\ell/\nu + (\Delta\boldsymbol{x})_{\ell+1} - \nu(\Delta\boldsymbol{x})_\ell + \nu^2(\Delta\boldsymbol{x})_{\ell+1}) = \frac{w}{8}\left(\nu+\frac{1}{\nu}\right)(\nu(\Delta\boldsymbol{x})_{\ell+1} - (\Delta\boldsymbol{x})_\ell). \end{aligned}$$

Since $x + 1/x$ is increasing on $[1,\infty)$, we get from (B.3),

$$\frac{w}{8}\left(\nu+\frac{1}{\nu}\right) \le \frac{w}{8}\left(R+\frac{1}{R}\right) \le 1,$$

and therefore

$$\begin{aligned}
\max\left(\frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell+2}},\ \frac{(\Delta\tilde{\boldsymbol{x}})_{2\ell+2}}{(\Delta\tilde{\boldsymbol{x}})_{2\ell+1}}\right) &= \max\left(\frac{v_\ell+\nu v_{\ell+1}+(\Delta\boldsymbol{x})_\ell-\nu v_{\ell+1}}{(\Delta\boldsymbol{x})_{\ell+1}-v_{\ell+1}},\ \frac{-(\nu v_\ell+v_{\ell+1})+(\Delta\boldsymbol{x})_{\ell+1}+\nu v_\ell}{(\Delta\boldsymbol{x})_\ell+v_\ell}\right)\\
&\leq \max\left(\frac{\nu(\Delta\boldsymbol{x})_{\ell+1}-\nu v_{\ell+1}}{(\Delta\boldsymbol{x})_{\ell+1}-v_{\ell+1}},\ \frac{\nu(\Delta\boldsymbol{x})_\ell+\nu v_\ell}{(\Delta\boldsymbol{x})_\ell+v_\ell}\right)=\nu.
\end{aligned}$$

This shows (B.2) for k odd and thereby the whole proposition. ☐

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