WAVELET SAMPLING TECHNIQUES*

Wim Sweldens†
Katholieke Universiteit Leuven, Belgium and University of South Carolina
and
Robert Piessens, Katholieke Universiteit Leuven, Belgium

KEY WORDS: wavelet, sampling, interpolation, quadrature

Abstract

In this paper we present several techniques to calculate the wavelet coefficients of a function from its samples. Interpolation, quadrature formulae and filtering methods are discussed and compared.

1 Introduction

1.1 Multiresolution analysis

We will first briefly review wavelets and multiresolution analysis. For more detailed treatments, one can consult [9, 15, 24, 26, 28].

A *multiresolution analysis* of $\mathrm{L}^2(\mathbb{R})$ is defined as a set of closed subspaces V_j with $j \in \mathbb{Z}$ that exhibit the following properties:

1. $V_j \subset V_{j+1}$,
2. $v(x) \in V_j \Leftrightarrow v(2x) \in V_{j+1}$,
3. $v(x) \in V_0 \Leftrightarrow v(x+1) \in V_0$,
4. $\bigcup_{j=-\infty}^{+\infty} V_j$ is dense in $\mathrm{L}^2(\mathbb{R})$,
5. $\bigcap_{j=-\infty}^{+\infty} V_j = \{\mathbf{0}\}$,
6. A *scaling function* $\varphi(x) \in V_0$ exists such that the set $\{\varphi(x \Leftrightarrow l) \mid l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

As a result a sequence $\{h_k\}$, exists such that the scaling function satisfies a *refinement equation*

$$\varphi(x) = 2 \sum_k h_k \, \varphi(2x \Leftrightarrow k). \tag{1}$$

The set of functions $\{\varphi_{j,l}(x) \mid l \in \mathbb{Z}\}$ with

$$\varphi_{j,l}(x) = \sqrt{2^j}\, \varphi(2^j x \Leftrightarrow l),$$

is a Riesz basis of V_j. Define now W_j as a complementary space of V_j in V_{j+1}, such that $V_{j+1} = V_j \oplus W_j$, $v(x) \in W_j \Leftrightarrow v(2x) \in W_{j+1}$, and $v(x) \in W_0 \Leftrightarrow v(x+1) \in W_0$. Consequently

$$\bigoplus_{j=-\infty}^{+\infty} W_j = \mathrm{L}^2(\mathbb{R}).$$

A function $\psi(x)$ is a *wavelet* if the set of functions $\{\psi(x \Leftrightarrow l) \mid l \in \mathbb{Z}\}$ is a Riesz basis of W_0. Since the wavelet is also an element of V_0, a sequence $\{g_k\}$ exists such that

$$\psi(x) = 2 \sum_k g_k \, \varphi(2x \Leftrightarrow k). \tag{2}$$

The set of wavelet functions $\{\psi_{j,l}(x) \mid l, j \in \mathbb{Z}\}$ is now a Riesz basis of $\mathrm{L}^2(\mathbb{R})$. The coefficients in the expansion of a function in the wavelet basis are given by the inner product with dual wavelets $\widetilde{\psi}_{j,l}(x) = \sqrt{2^j}\widetilde{\psi}(2^j x \Leftrightarrow l)$ such that

$$f(x) = \sum_{j,l} \langle f, \widetilde{\psi}_{j,l} \rangle \, \psi_{j,l}(x).$$

Likewise a projection on V_j is given by

$$P_j f(x) = \sum_l \langle f, \widetilde{\varphi}_{j,l} \rangle \, \varphi_{j,l}(x)$$

---

* To appear in the proceedings of the Joint Statistical Meetings, San Francisco, August 1993.
† Research Assistant of the National Fund of Scientific Research Belgium and partially supported by ONR Grant N00014-90-J-1343.

where $\widetilde{\varphi}_{j,l}(x) = \sqrt{2^j}\widetilde{\psi}(2^j x \Leftrightarrow l)$ are the dual scaling functions. The dual functions have to satisfy the biorthogonality conditions

$$\langle \varphi_{j,l}, \tilde{\varphi}_{j,l'} \rangle = \delta_{l-l'}$$

and

$$\langle \psi_{j,l}, \tilde{\psi}_{j',l'} \rangle = \delta_{j-j'}\delta_{l-l'}.$$

They satisfy refinement relations similar to (1) and (2) involving sequences $\{\tilde{h}_k\}$ and $\{\tilde{g}_k\}$. In case the basis functions coincide with their duals, the basis is orthogonal.

1.2 Fourier analysis

Recall the following definitions. The Fourier transform of a function $f(x)$ is given by

$$\hat{f}(\omega) = \int_{-\infty}^{+\infty} f(x)\, e^{-i\omega x}\, dx.$$

The discrete Fourier transform (dft) of a sequence $\{s_k\}$ is defined as

$$\hat{s}(\omega) = \sum_l s_l\, e^{-i\omega l}.$$

In case a finite number of coefficients s_k is non zero, $\hat{s}(\omega)$ is a trigonometric polynomial. Finally the Zak transform of a function is defined as [23],

$$\mathcal{Z}f(x,\omega) = \sum_l f(x+l)\, e^{-i\omega l},$$

and the Poisson summation formula is given by

$$\sum_l f(x+l) = \sum_k \hat{f}(2\pi k)\, e^{i2\pi kx}.$$

Taking the Fourier transform of equations (1) and (2) now yields

$$\hat{\varphi}(\omega) = \hat{h}(\omega/2)\, \hat{\varphi}(\omega/2) \tag{3}$$

and

$$\hat{\psi}(\omega) = \hat{g}(\omega/2)\, \hat{\psi}(\omega/2). \tag{4}$$

Similar definitions and equations hold for the dual functions. A necessary condition for biorthogonality is then that $\forall \omega \in \mathbb{R}$,

$$\begin{aligned} \overline{\hat{h}(\omega)}\,\hat{\tilde{h}}(\omega) + \overline{\hat{g}(\omega)}\,\hat{\tilde{g}}(\omega) &= 1 \\ \overline{\hat{h}(\omega)}\,\hat{\tilde{h}}(\omega+\pi) + \overline{\hat{g}(\omega)}\,\hat{\tilde{g}}(\omega+\pi) &= 0. \end{aligned} \tag{5}$$

A classical example of a scaling function is the polynomial B-spline of order m, where

$$\hat{\varphi}(\omega) = \left(\frac{1 \Leftrightarrow e^{-i\omega}}{i\omega}\right)^m.$$

1.3 Wavelets and polynomials

The moments of the scaling function and wavelet are defined as ($p \in \mathbb{N}$)

$$\mathcal{M}_p = \int_{-\infty}^{+\infty} x^p\, \varphi(x)\, dx$$

and

$$\mathcal{N}_p = \int_{-\infty}^{+\infty} x^p\, \psi(x)\, dx.$$

For the moments of the dual functions we use the notations $\widetilde{\mathcal{M}}_p$ and $\widetilde{\mathcal{N}}_p$. The scaling functions cannot have a vanishing integral and are normalized with $\mathcal{M}_0 = \widetilde{\mathcal{M}}_0 = 1$. The number of vanishing dual wavelet moments is denoted by N, and the number of vanishing wavelet moments by $\widetilde{N}$. These numbers are also equal to the multiplicity of 0 as a root of respectively $\hat{\tilde{g}}(\omega)$ or $\hat{g}(\omega)$ and, using (5), also to the multiplicity of π as a root of respectively $\hat{h}(\omega)$ or $\hat{\tilde{h}}(\omega)$. It follows from (3) that $\hat{\varphi}(\omega)$ and its first $N \Leftrightarrow 1$ derivatives vanish at $2k\pi$, $k \neq 0$, or

$$i^p\, \hat{\varphi}^{(p)}(2k\pi) = \mathcal{M}_p\, \delta_k \quad \text{for} \quad 0 \leqslant p < N. \tag{6}$$

Using the Poisson summation formula, this is equivalent with

$$\sum_l (x \Leftrightarrow l)^p \varphi(x \Leftrightarrow l) = \mathcal{M}_p \quad \text{for} \quad 0 \leqslant p < N. \tag{7}$$

This implies that any polynomial with degree less than N can be written as a linear combination of the functions $\varphi_{j,l}(x)$ with $l \in \mathbb{Z}$. The condition (6), known as the Strang-Fix condition, [21, 34, 35], implies that if $f \in \mathcal{C}^N$ then ($h = 2^{-n}$)

$$\langle f, \psi_{n,l} \rangle = \mathcal{O}(h^{N+1/2})$$

and

$$\|f(x) \Leftrightarrow P_n f(x)\| = \mathcal{O}(h^N). \tag{8}$$

These properties make wavelets suited for applications such as data compression.

1.4 Fast wavelet transform

Given the coefficients $\nu_{j,l} = \langle f, \widetilde{\varphi}_{j,l} \rangle$ of a function in V_j, one can use the refinement relations to find its coefficients in the bases of the spaces V_{j-1} and W_{j-1},

$$\nu_{j-1,l} = \sqrt{2} \sum_k \tilde{h}_{k-2l}\, \nu_{j,k},$$

and

$$\mu_{j-1,l} = \langle f, \widetilde{\psi}_{j-1,l} \rangle = \sqrt{2} \sum_k \tilde{g}_{k-2l}\, \nu_{j,k}.$$

The inverse step is given by,

$$\nu_{j,k} = \sqrt{2} \sum_l h_{k-2l}\, \nu_{j-1,l} + \sqrt{2} \sum_l g_{k-2l}\, \mu_{j-1,l}.$$

When applied recursively, these formulae define a transformation, the *fast wavelet transform* [26, 27]. This transform can be seen as a filtering operation which involves the filters $\hat{h}$, $\hat{\tilde{h}}$, $\hat{g}$, and $\hat{\tilde{g}}$. The first two are low pass filters, the last two band pass filters. In case the scaling function and wavelet are compactly supported, these filters are finite impulse response filters and a fast and accurate numerical implementation is possible.

1.5 Contents of the paper

The theory of multiresolution analysis tells us that we can find the coefficients of the expansion of a function in the wavelet basis as inner products with the dual wavelets. In practise one will first calculate the projection $P_n f(x)$ for a fixed n, and then use the fast wavelet transform to find the wavelet coefficients on the coarser levels, i.e. the $\mu_{j,l}$ with $j < n$.

In many applications we do not know $f(x)$ as a function of a continuous variable, but only a sequence of data $\{d_k\}$. We assume that the data are uniform samples of the function $f(x)$. In this paper we study computational techniques to find the coefficients of an initial approximation in V_n from these data. Without loss of generality we consider the case where $n = 0$ is the finest level and assume that the sampling distance is 1. We write the initial approximation in V_0 as

$$v(x) = \sum_l \nu_l\, \varphi(x \Leftrightarrow l).$$

It is clear the the ν_l should be linear functionals of the data $\{d_k\}$, We consider several approaches and discuss in which applications they can be used. First we study interpolation methods. In a second section we consider schemes based on numerical integration. Finally we mention some methods used in signal processing.

2 Interpolation

2.1 The general case

We associate the samples with the integer locations and look for the function $v \in V_0$ that interpolates the data sequence, i.e. $v(k) = d_k$. We first consider the trivial case.

Definition 1 *A scaling function $\varphi(x)$ is interpolating if $\varphi(k) = \delta_k$.*

In this case the solution is immediately given by $\nu_l = d_l$. The following lemma then follows from the refinement relation:

Lemma 2 *If $\varphi(x)$ is interpolating, then $h_{2k} = \varphi(k)/2 = \delta_k/2$.*

In general the problem can be written as

$$d_k = \sum_l \nu_l\, \varphi(k \Leftrightarrow l), \tag{9}$$

or

$$\{d_k\} = \{\nu_k\} * \{\varphi(k)\},$$

where $*$ stands for convolution. This shows that the solution can be found be inverting an infinite Toeplitz matrix which is banded in case the scaling function is compactly supported. Let $\hat{d}(\omega)$, $\hat{\nu}(\omega)$ and $\hat{p}(\omega)$ be the dft's of the sequences $\{d_k\}$, $\{\nu_k\}$ and $\{\varphi(k)\}$. The relationship (9) can then be written as

$$\hat{d}(\omega) = \hat{\nu}(\omega)\, \hat{p}(\omega).$$

This leads to the following result:

Lemma 3 *The interpolation problem (9) has a unique solution if $\hat{p}(\omega)$ does not vanish.*

This technique was studied in [1, 3, 37]. A problem here is that $1/\hat{p}(\omega)$ is not a trigonometric polynomial and that thus each coefficient ν_l depends on all the data samples d_k. This is evidently not very useful computation wise. It is however possible to show that the coefficients of $1/\hat{p}(\omega)$ decrease exponentially. Consequently, if one sets forth a certain numerical accuracy, the infinite convolution can be broken off.

We can use this result to construct an interpolating scaling function which generates V_0 as

$$\hat{\varphi}_{interpol}(\omega) = \frac{\hat{\varphi}(\omega)}{\hat{p}(\omega)}.$$

Again, even if φ is compactly supported, the interpolation function generally is not but instead it is exponentially decreasing. A typical example are the cardinal spline interpolants of even order [31]. One exception is second order where the B-spline itself (the so called hat function) is interpolating.

2.2 Shifted interpolation

In case $\hat{p}(\omega)$ vanishes an interpolating function does not exist. We can then add some flexibility to the interpolation problem by not necessarily associating the data with the integer locations but allowing a shift $\tau \in (0, 1)$ and formulate the problem as

$$d_k = \sum_l \nu_l \, \varphi(k \Leftrightarrow l + \tau).$$

We would have an extremely nice situation if shifting the scaling function would yield an interpolating function. The following lemma then tells us what the shift then should be.

Lemma 4 *If $\varphi(\tau + k) = \delta_k$ and $N \geqslant 1$, then $\tau = \mathcal{M}_1$.*

Proof. Follows immediately from the fact that

$$\sum_l (x \Leftrightarrow l) \, \varphi(x \Leftrightarrow l) = \mathcal{M}_1.$$

□

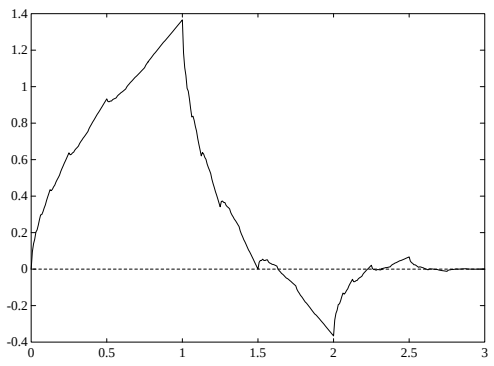

Figure 1: Daubechies' orthogonal scaling function with $N = 2$.

Unfortunately this property is hardly ever satisfied. It was remarked by Mary Ellen Bock that in case of the orthogonal Daubechies' scaling functions constructed in [14], it is almost satisfied. For the scaling function with $N = 2$, which has support $[0, 3]$, one can check that $\mathcal{M}_1 = (1 + \sqrt{3})/4 \approx 0.683$, $\varphi(\mathcal{M}_1) \approx 1.00020859077$, $\varphi(\mathcal{M}_1 + 1) \approx \Leftrightarrow 4.17181539384E \Leftrightarrow 04$, and $\varphi(\mathcal{M}_1 + 2) \approx 2.08590769692E \Leftrightarrow 04$. This function is shown in figure 1. In this case one can construct a very elegant geometric interpretation of the fast wavelet transform [6].

In the general case the solution to the shifted interpolating problem is given by [22]

$$\hat{\nu}(\omega) = \frac{\hat{d}(\omega)}{\mathcal{Z}\varphi(\tau, \omega)},$$

provided that the denominator does not vanish. It is almost always possible to find a τ such that $\mathcal{Z}\varphi(\tau, \omega)$ does not vanish. Note that $\mathcal{Z}\varphi(\tau, \omega)$ is a 2π periodic function in ω.

The shifted interpolation is particularly useful when the scaling function is symmetric around a non integer. A typical example are the B-splines of odd order, which are symmetric around an integer + 1/2, such that one can take $\tau = 1/2$. In [22] more general criteria for the choice of τ are studied.

2.3 Compactly supported interpolating scaling functions

An interpolation scaling function can also be constructed from a pair of biorthogonal scaling functions as

$$\Phi(x) = \int_{-\infty}^{+\infty} \varphi(y+x)\,\overline{\widetilde{\varphi}(y)}\,dy.$$

The interpolation property immediately follows from the biorthogonality condition. If the scaling function and its dual are compactly supported, so is the interpolation function. In the case of an orthogonal scaling function the interpolating function is just its autocorrelation function. The interpolating function and its translates do not generate the same space as φ and its translates.

It is easy to see that the interpolating function satisfies a refinement relation with coefficients $H_k = \Phi(k/2)/2$ and where

$$\hat{H}(\omega) = \hat{h}(\omega)\,\overline{\hat{\widetilde{h}}(\omega)}.$$

The interpolating scaling function has several nice properties. It is smoother then φ and $\widetilde{\varphi}$, it is symmetric and it can reproduce the polynomials with degree less than $N + \widetilde{N}$. One can build a multiresolution analysis where the dual of the interpolating scaling function is formally the Dirac function such that $\hat{\widetilde{H}}(\omega) = 1$. It is possible to show that a wavelet function which generates complementary spaces W_j can be chosen as

$$\Psi(x) = \Phi(2x \Leftrightarrow 1).$$

The dual wavelet is then a linear combination of Dirac impulses and has $N + \widetilde{N}$ vanishing moments or, more precisely,

$$\hat{\widetilde{\Psi}}(\omega) = \Leftrightarrow e^{-i\omega}\overline{H(\omega+\pi)}.$$

Evidently this yields only a multiresolution analysis for function spaces where pointwise evaluation is a bounded operator. This means we need to impose some smoothness on the functions.

A fast wavelet transform with finite impulse response filters follows immediately from this construction. A disadvantage is that these filters introduce considerable aliasing in the fast wavelet transform.

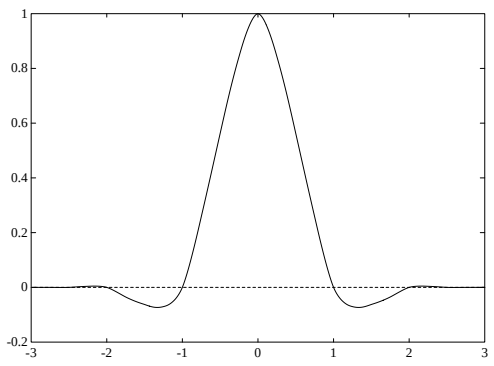

Figure 2: Interpolation scaling function

Recently Lemarié [25], Shensa [33], and Beylkin and Saito [30] noted that this construction, started from the Daubechies orthogonal wavelets, yields a family of interpolating functions which were originally studied by Deslauriers and Dubuc in [17, 18]. They were used for the characterization of function spaces in [20] and in signal processing applications in [30]. The interpolating scaling function constructed as the autocorrelation function of the Daubechies' orthogonal scaling function with $N = 2$ is shown in figure 2. In fact one get exactly the same function starting from any pair of compactly supported biorthogonal scaling functions from [11] with $N + \widetilde{N} = 4$.

These interpolation schemes are also closely related to stationary subdivision see e.g. [7].

3 Quadrature formulae

3.1 General idea

In this section we study schemes to find the ν_l by numerically approximating the inner products $\langle f, \widetilde{\varphi}_{0,l} \rangle$ using a quadrature formula. A quadrature formula is determined by its *weights* w_k and *abscissae* x_k such that

$$\int_{-\infty}^{+\infty} f(x)\,\overline{\widetilde{\varphi}(x)}\,dx \;\approx\; Q[f(x)] = \sum_{k=0}^{r} w_k\, f(x_k). \tag{10}$$

In order to let $f(x_k)$ correspond to the data, the abscissae x_k have to be chosen as $k+\tau$. The coefficients are then given by

$$\nu_l = \sum_{k=0}^{r} w_{k-l}\, d_k.$$

The unknowns are now the shift τ and the weights w_k.

A popular technique to solve problems in numerical analysis is to design an approximate solution scheme which is exact for polynomials. This method then usually yields acceptable results for smooth functions since they locally resemble polynomials. When numerically approximating integrals, this leads to the following definition.

Definition 5 *The* degree of accuracy *of a quadrature formula is q if it yields the exact result for every polynomial of degree less than or equal to q.*

The degree of accuracy determines the asymptotic convergence order as follows: if $f(x)$ belongs to $\mathcal{C}^{q+1}$, then ($h = 2^{-n}$)

$$\frac{\nu_{n,l} \Leftrightarrow 2^{-n/2} Q[f(2^{-n}(x+l))]}{\nu_{n,l}} = \mathcal{O}(h^{q+1}). \tag{11}$$

This can easily be seen using the Taylor expansion.

The number of abscissae r determines the efficiency of a quadrature formula since the number of function evaluations and algebraic operations is proportional to r. The quadrature formula is usually constructed by demanding that

$$Q[x^i] = \widetilde{\mathcal{M}}_i \quad \text{for} \quad 0 \leqslant i \leqslant q,$$

which leads to an algebraic system.

Comparing equations (8) and (11), we see that the degree of accuracy should be at least equal to $N \Leftrightarrow 1$, otherwise the quadrature formula will ruin the asymptotic convergence order of the wavelet approximation. Quadrature formula for use in connection with multiresolution analysis were studied in [5, 36].

3.2 Trapezoidal rule

A simple quadrature formula is the *trapezoidal rule*, where

$$Q[f(x)] = \sum_k \widetilde{\varphi}(k)\, f(k). \tag{12}$$

In general the application of this rule is limited because it only has a degree of accuracy equal to one but here the following lemma holds:

Lemma 6 *If the scaling function satisfies the Strang-Fix condition (6), the degree of accuracy of the trapezoidal rule (12) is equal to $\widetilde{N} \Leftrightarrow 1$.*

This easily seen from equation (7) for $x = 0$. The trapezoidal rule can thus be used in a multiresolution analysis without ruining the asymptotical convergence order if $\widetilde{N} \geqslant N$. However in general it is not very efficient. In case $\varphi(x)$ is not compactly supported, the sum in (12) has to be broken of which usually leads to a large number of abscissae. But also when $\varphi(x)$ is compactly supported, its efficiency is low: the Daubechies' orthogonal scaling functions have a support length of $2N \Leftrightarrow 1$, such that $r = 2N \Leftrightarrow 2 = 2q$ while even with a fixed value of τ one can achieve $r = q+1$. Only in the case of cardinal B-splines, the trapezoidal rule is useful because here $r = N \Leftrightarrow 1 = q$.

3.3 One point formula

Since the integral of the scaling function is 1, we can write a one point formula ($r = 1$) as $Q[f(x)] = f(\tau)$. Evidently, if $\tau = \mathcal{M}_1$, the degree of accuracy is equal to 1. In the case of orthogonal wavelets, the following lemma holds [36]:

Lemma 7 *If $\varphi(x)$ is an orthogonal scaling function with $N > 1$, then the one point quadrature formula has a degree of accuracy equal to 2.*

Consequently it can be used in case $N \leqslant 3$ without ruining the convergence rate.

In [15, 16] Ingrid Daubechies constructed orthogonal scaling functions with compact support that have $N \Leftrightarrow 1$ vanishing moments,

$$\mathcal{M}_p = 0 \quad \text{for} \quad 1 \leqslant p < N, \tag{13}$$

where again N is the number of vanishing wavelet moments. These wavelets were called *coiflets* after Ronald Coifman who asked for their construction. We see from (7) that they also satisfy

$$\sum_k k^p \varphi(k) = \delta_p \quad \text{for} \quad 0 \leqslant p < N. \tag{14}$$

In this case the one point quadrature formula with $\tau = 0$ immediately has a degree of accuracy of $N \Leftrightarrow 1$. This formula was used in numerical analysis applications in [5].

In the case of a multiple point formula, a general numerical scheme to find the shift and weights is described in [36].

3.4 Remarks

When using a quadrature formula we get the exact projection $P_0 f(x)$ in case f is a polynomial of degree less than or equal to the degree of accuracy. Moreover in case its degree is less than N we will recover the function exactly since $P_0 f(x) = f(x)$. This means we can reconstruct a polynomial from its samples. Such a method is usually referred to a as *quasi-interpolating scheme* [9]. In case the scaling function is compactly supported this property holds also locally.

Evidently the asymptotic error estimates are only useful whenever one has the opportunity of increasing the sampling rate and when the underlying function is smooth. This is often the case in numerical analysis problems such as the solution of differential equations. In applications such as signal processing the sampling rate is usually fixed and the asymptotical estimates are not useful.

4 Signal processing approach

From equation (6) one can understand that the functions of V_0 have most of their energy in the frequency band $[\Leftrightarrow\pi, \pi]$. In fact in case the scaling function is taken to be the Shannon sampling function,

$$\varphi(x) = \frac{\sin(\pi x)}{\pi x},$$

then this is exactly the case. Note that this function is an orthogonal and interpolating scaling function, but that it dies off very slow.

The two techniques we studied so far can be treated in a unified manner. We know that the coefficients ν_l are linear functionals of the data d_k and that this relationship is invariant to integer translates. This implies that every solution can be written as a convolution [33]. In both cases we allowed the samples to correspond with function values on a shifted grid, $d_k = f(\tau + k)$. The general form of the solution involves then a sequence $\{a_k\}$ such that

$$\nu_l = \sum_k a_{l-k} f(k + \tau).$$

Because of the shift invariance we can concentrate on the case $l = 0$ where

$$\nu_0 = \frac{1}{2\pi} \langle \hat{f}, A \rangle,$$

and $A(\omega)$ is a 2π-periodic function,

$$A(\omega) = \overline{\hat{a}(\omega)}\, e^{-i\omega\tau}.$$

As we know from the Parseval identity, the exact value of ν_0 is given by

$$\nu_0 = \frac{1}{2\pi} \langle \hat{f}, \hat{\tilde{\varphi}} \rangle.$$

We can now formulate the problem as follows: find the unknowns τ and a_k such that $A(\omega)$ in some sense is a good approximation for $\hat{\tilde{\varphi}}(\omega)$ in the neighborhood of the interval $[\Leftrightarrow\pi, \pi]$.

One can consider several solutions

1. The simplest solution just lets $\hat{a}(\omega) = 1$ which corresponds to $\nu_l = a_l$. This means that $A(0) = \hat{\tilde{\varphi}}(0)$. To let the phase correspond at $\omega = 0$, one need to chose $\tau = \widetilde{\mathcal{M}}_1$. This corresponds to the one point quadrature formula.

2. The construction of a general quadrature formula lets the derivatives of $A(\omega)$ and $\hat{\tilde{\varphi}}(\omega)$ at $\omega = 0$ coincide up to the order equal to the degree of accuracy.

3. One can let A coincide with the 2π periodization of $\hat{\tilde{\varphi}}$,

$$A(\omega) = \sum_k \hat{\tilde{\varphi}}(\omega + k2\pi).$$

Using the Poisson summation formula we see that

$$A(\omega) = \sum_l \tilde{\varphi}(l)\, e^{-i\omega l}.$$

This thus corresponds to the use of the trapezoidal rule as a quadrature formula.

4. One can let $A(\omega)$ coincide with $\hat{\tilde{\varphi}}(\omega)$ on the interval $[\Leftrightarrow\pi, \pi]$. This will lead to the exact result in case the function f is band limited. A sampling theorem similar to the classical Shannon sampling theorem can then be proven [2]. Note that $A(\omega)$ is not a trigonometric polynomial such that the scheme will be non local. The decay of the coefficients depends on the smoothness of $A(\omega)$ as a 2π periodic function. The decay is here a little better than in the classical Shannon case because $\hat{\tilde{\varphi}}(\omega)$ has a root of multiplicity $\tilde{N}$ at π and $\Leftrightarrow\pi$.

5. Remember the interpolation solution corresponds to choosing

$$A(\omega) = 1/\overline{\hat{p}(\omega)}.$$

It is interesting to remark that in the orhtogonal case this is almost exactly the inverse of the solution given by the trapezoidal rule.

6. Evidently many other criteria and corresponding solutions can be suggested. One possibility would be to fix the degree of the trigonometric polynomial $A(\omega)$ and then look for the minimax approximation of $\hat{\tilde{\varphi}}(\omega)$ in the interval $[\Leftrightarrow\pi, \pi]$.

5 Wavelets on an interval

So far our discussion only involved the case of the real line which is invariant for integer shifts. Recently several constructions of wavelet on an interval became available [4, 8, 12, 13, 29]. These constructions all have in common that the functions that are supported, in some sense, away from the endpoints correspond to the ones from the real line, while new basis functions are constructed near the boundary. One of the problems is that the shift invariance is lost at the boundary. Therefore it is not immediately clear how the coefficients should correspond to the data. Fourier techniques cannot be used any more.

In [13] a so called *preconditioning* step is introduced. This is inspired by the fact that the coefficients of a polynomial in the V_0 space are not a polynomial sequence any more. The preconditioning step involves applying a linear transform to the data samples near the boundary such that in case of a polynomial data sequence one gets the coefficients of a polynomial in the V_0 space. This assures that smooth sequences will have small wavelet coefficients which is one of the basic reasons why wavelets are suited for data compression.

In [4] so called recursive wavelets are introduced. We explain the idea first on the real line. The V_0 space is here defined as the space of the functions that are piecewise constant on the intervals $[k, k+1)$. Each data sequence then corresponds to a function in V_0 by letting

$$v(x) = \sum_k d_k \chi_{[0,1)}(x \Leftrightarrow k),$$

where χ is the indicator function. The V_0 space is thus generated by "block" functions $\varphi_{0,l}$ which are orthogonal. For $j < 0$ we define the basis functions for V_j and W_j recursively through the relations

$$\varphi_{j,k} = \sum_k h_{m-2k}\, \varphi_{j+1,m},$$

and

$$\psi^j_{j,k} = \sum_k g_{m-2k}\, \varphi_{j+1,m},$$

and similarly for the dual basis functions. This assures that the wavelet coefficients can be calculated with the fast wavelet transform. Note that the wavelets here are not any longer the dilates and translates of one particular function.

When applying the above construction for wavelets on a closed interval we obtain recursive wavelets for the interval. This construction has the advantage that sampling and preconditioning on the finest level become trivial.

Note that the recursive wavelets can also be used in case the d_k are not the samples of the function f but rather the inner products of f with the block function $\chi_{[k,k+1)}$. How one can construct a multiresolution analysis from these coefficients is also discussed in [19].

Acknowledgement

The first named author is Research Assistant of the National Fund of Scientific Research Belgium and was partially supported by ONR Grant N00014-90-J-1343. He also would like to thank Mary Ellen Bock and Ingrid Daubechies for interesting and stimulating discussions.

References

[1] A. Aldroubi and M. Unser. Families of wavelet transforms in connection with Shannon's sampling theory and the Gabor transform. In [10], pages 509–528.

[2] A. Aldroubi and M. Unser. Sampling procedures in function spaces and asymptotic equivalence with Shannon's sampling theory. *Numer. Funct. Anal. Optim.*, to appear.

[3] A. Aldroubi and M. Unser. Families of multiresolution and wavelet spaces with optimal properties. *Numer. Funct. Anal. Optim.*, 14:417–446, 1993.

[4] L. Andersson, N. Hall, B. Jawerth, and G. Peters. Wavelets on closed subsets of the real line. In [32], pages 1–61.

[5] G. Beylkin, R. Coifman, and V. Rokhlin. Fast wavelet transforms and numerical algorithms i. *Comm. Pure Appl. Math.*, 44:141–183, 1991.

[6] M. E. Bock and G. Pliego. Estimating functions with wavelets part ii: Using a Daubechies wavelet in nonparametric regression. *Statistical Computing and Statistical Graphics Newsletter*, 3(2):27–34, November 1992.

[7] A. S. Cavaretta, W. Dahmen, and C. A. Micchelli. Stationary subdivision. *Memoirs Amer. Math. Soc.*, 93(453), 1991.

[8] C. Chui and E. Quak. Wavelets on a bounded interval. In D. Braess and L. L. Schumaker, editors, *Numerical Methods of Approximation Theory*, pages 1–24. Birkhäuser-Verlag, Basel, 1992.

[9] C. K. Chui. *An Introduction to Wavelets.* Academic Press, San Diego, CA, 1992.

[10] C. K. Chui, editor. *Wavelets: A Tutorial in Theory and Applications.* Academic Press, San Diego, CA, 1992.

[11] A. Cohen, I. Daubechies, and J. Feauveau. Bi-orthogonal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 45:485–560, 1992.

[12] A. Cohen, I. Daubechies, B. Jawerth, and P. Vial. Multiresolution analysis, wavelets and fast algorithms on an interval. *C. R. Acad. Sci. Paris Sér. I Math.*, I(316):417–421, 1993.

[13] A. Cohen, I. Daubechies, and P. Vial. Multiresolution analysis, wavelets and fast algorithms on an interval. *Appl. Comput. Harmon. Anal.*, 1(1):54–81, 1993.

[14] I. Daubechies. Orthonormal bases of compactly supported wavelets. *Comm. Pure Appl. Math.*, 41:909–996, 1988.

[15] I. Daubechies. *Ten Lectures on Wavelets.* CBMS-NSF Regional Conf. Series in Appl. Math., Vol. 61. Society for Industrial and Applied Mathematics, Philadelphia, PA, 1992.

[16] I. Daubechies. Orthonormal bases of compactly supported wavelets II: Variations on a theme. *SIAM J. Math. Anal.*, 24(2):499–519, 1993.

[17] G. Deslauriers and S. Dubuc. Interpolation dyadique. In *Fractals, dimensions non entières et applications*, pages 44–55. Masson, Paris, 1987.

[18] G. Deslauriers and S. Dubuc. Symmetric iterative interpolation processes. *Constr. Approx.*, 5(1):49–68, 1989.

[19] D. L. Donoho. Smooth wavelet decompositions with blocky coefficient kernels. In [32], pages 259–308.

[20] D. L. Donoho. Interpolating wavelet transforms. Preprint, Department of Statistics, Stanford University, 1992.

[21] G. Fix and G. Strang. Fourier analysis of the finite element method in Ritz–Galerkin theory. *Stud. Appl. Math*, 48:265–273, 1969.

[22] A. J. E. M. Janssen. The Zak transform and sampling theorems for wavelet subspaces. Preprint, Philips Research Laboratories, Eindhoven, The Netherlands.

[23] A. J. E. M. Janssen. The Zak transform: a signal transform for sampled time-continuous signals. *Philips J. Res.*, 43:23–69, 1988.

[24] B. Jawerth and W. Sweldens. An overview of wavelet based multiresolution analyses. *SIAM Rev.*, 36(3):377–412, 1994.

[25] P. G. Lemarié. Some remarks on wavelet theory and interpolation. Preprint, Mathématiques, Paris XI, Orsay, 1991.

[26] S. G. Mallat. Multifrequency channel decompositions of images and wavelet models. *IEEE Trans. Acoust. Speech Signal Process.*, 37(12):2091–2110, 1989.

[27] S. G. Mallat. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbb{R})$. *Trans. Amer. Math. Soc.*, 315(1):69–87, 1989.

[28] Y. Meyer. *Ondelettes et Opérateurs,* I: *Ondelettes,* II: *Opérateurs de Calderón-Zygmund,*
III: (with R. Coifman), *Opérateurs multilinéaires.* Hermann, Paris, 1990. English translation of first volume, *Wavelets and Operators*, is published by Cambridge University Press, 1993.

[29] Y. Meyer. Ondelettes sur l'intervalle. *Rev. Mat. Iberoamericana*, 7:115–133, 1992.

[30] N. Saito and G. Beylkin. Multiresolution representations using the auto-correlation functions of compactly supported wavelets. *IEEE Trans. Signal Process.*, 41(12):3584–3590, 1993.

[31] I. J. Schoenberg. Cardinal interpolation and spline functions. *J. Approx. Theory*, 2:167–627, 1969.

[32] L. L. Schumaker and G. Webb, editors. *Recent Advances in Wavelet Analysis.* Academic Press, New York, 1993.

[33] M. J. Shensa. Wedding the à trous and Mallat algorithms. *IEEE Trans. Signal Process.*, 40(10):2464–2482, 1992.

[34] G. Strang. Wavelets and dilation equations: A brief introduction. *SIAM Rev.*, 31(4):614–627, 1989.

[35] G. Strang and G. Fix. A Fourier analysis of the finite element variational method. In *Constructive aspects of Functional Analysis.* Edizione Cremonese, Rome, 1973.

[36] W. Sweldens and R. Piessens. Quadrature formulae and asymptotic error expansions for wavelet approximations of smooth functions. *SIAM J. Numer. Anal.*, 31(4):1240–1264, 1994.

[37] G. G. Walter. A sampling theorem for wavelet subspaces. *IEEE Trans. Inform. Theory*, 38:881–884, 1992.

University of South Carolina, Department of Mathematics, Columbia SC 29201, sweldens@math.scarolina.edu.
Katholieke Universiteit Leuven, Department of Computer Science, Celestijnenlaan 200 A, B 3001 Leuven, Belgium