Wavelet probing for compression based segmentation[1]

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ABSTRACT

In this paper we show how wavelets can be used for data segmentation. The basic idea is to split the data into smooth segments that can be compressed separately. A fast algorithm that uses wavelets on closed sets and wavelet probing is presented.

1 INTRODUCTION

Segmentation of data is important in many aspects of image and signal processing. The general idea is to divide the data into segments such that each segment can be treated separately to achieve specific goals such as pattern recognition, feature extraction, data compression, template matching, noise reduction, etc. The criterion to determine the splitting locations evidently depends on the application.

In this paper we consider compression based segmentation. As we know, smooth data can be compressed efficiently. In general data is only smooth locally and contains jump discontinuities, discontinuities in its derivative or sudden changes in frequency. These discontinuities usually are expensive to encode. This leads to artifacts, such as smoothing, at high compression ratios. If the locations of these discontinuities are used as splitting points, each segment will be smooth and can be compressed efficiently. The reconstructed data will still have the sharp localization of the discontinuities.

This idea can be used in two different ways for image compression. First the image can be segmented into smooth regions by indicating edges (which do not need to be closed) and compress each region separately. Secondly also these edges have to be coded efficiently. To compress the edge information, we can now use the same idea in one dimension on each edge curve. In this way we can avoid the blurring of edges which is a typical artifact that appears at high compression ratios.

Related ideas are presented in.[13,14,17] The algorithm presented here is based on a conjecture of David Marr that images can be coded by their multiscale edges.[18] The idea is that the multiscale edges of an image correspond

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to the local maxima of a wavelet transform. An iterative reconstruction algorithm is presented here.

In this paper we will focus on the one-dimensional case. Many segmentation algorithms have already been proposed. Two popular methods are k-curvature and arc-chord distance for corner detection.[12] Other methods are based on e.g. statistical models,[10,11] entropy,[24] or hierarchical representations.[2,3]

We propose a compression based segmentation algorithm that incorporates wavelets. The wavelet transform is closely related to multiscale analysis and has hierarchical structure, which is particularly suited for fast numerical implementation. Wavelet compression is based on the fact that the wavelet coefficients that correspond to smooth parts of the data become small on the finer levels and can be set to zero without distorting the original data too much. Wavelets were originally constructed for the whole real line. Our algorithm makes use of a new construction of wavelets on an interval. This construction provides a discrete wavelet transform of sequences of arbitrary length. Therefore it allows us to split a function around a certain point i.e. given the a function on an interval $[A, B]$, calculate its wavelet transform on the intervals $[A, C]$ and $[C, B]$. Our segmentation algorithm splits the data between every two samples and checks whether introducing a split point will facilitate compression. This is measured by a compression based cost function. The segmentation points are then chosen as the points where the cost function has a local minimum.

It is easy to understand that the splitting algorithm only needs to recalculate the coefficients of the wavelets which are supported in the neighborhood of the splitting point. As a result it is a typical example of *wavelet probing*.[1] Basically wavelet probing is any algorithm which involves wavelet coefficients over all the levels which correspond to a fixed location. Because of the compact support of the wavelets the work load of wavelet probing is proportional only to the number of levels. This means that probing at all locations is only a $\mathcal{O}(n\log(n))$ algorithm where n is the number of data points.

This paper is organized as follows. In section 2 we briefly review wavelets and multiresolution analysis. In section 3 we introduce wavelets on an closed interval and associated wavelet transform. Then in section 4 we describe splitting and merging algorithms. Segmentation based on wavelet compression will be discussed in section 5. Finally we give numerical results in section 6.

2 WAVELETS AND MULTIRESOLUTION ANALYSIS

We will briefly review wavelets and multiresolution analysis. For detailed treatments, we refer to.[4,8,15,19] A *multiresolution analysis* of $\mathrm{L}^2(\mathbb{R})$ is defined as a set of closed subspaces V_ν with $\nu \in \mathbb{Z}$ that exhibit the following properties:

1. $V_\nu \subset V_{\nu+1}$,
2. $v(x) \in V_\nu \Leftrightarrow v(2x) \in V_{\nu+1}$, and $v(x) \in V_0 \Leftrightarrow v(x+1) \in V_0$,
3. $\bigcup_{\nu=-\infty}^{+\infty} V_\nu$ is dense in $\mathrm{L}^2(\mathbb{R})$ and $\bigcap_{\nu=-\infty}^{+\infty} V_\nu = \{\mathbf{0}\}$,
4. A *scaling function* $\varphi(x) \in V_0$ exists such that the set $\{\varphi(x \Leftrightarrow l) \mid l \in \mathbb{Z}\}$ is a Riesz basis of V_0.

These properties imply the existence of a sequence $(h_k) \in \ell^2(\mathbb{Z})$ for which the scaling function satisfies the *refinement equation*

$$\varphi(x) \;=\; \sqrt{2}\sum_k h_k\, \varphi(2x \Leftrightarrow k). \tag{1}$$

It also follows that the functions $\varphi_{\nu,l}(x) = 2^{\nu/2}\,\varphi(2^\nu x \Leftrightarrow l), l \in \mathbb{Z}$, constitute a Riesz basis of V_ν. It is customary to introduce detail spaces W_ν, that are complement spaces of V_ν in $V_{\nu+1}$, i.e.,

$$V_{\nu+1} \;=\; V_\nu \;\dot{+}\; W_\nu, \tag{2}$$

and such that there exists a mother wavelet $\psi(x)$ with the property that the functions $\psi_{\nu,l}(x) = 2^{\nu/2}\,\psi(2^\nu x \Leftrightarrow l), l \in \mathbb{Z}$, span the space W_ν. It now follows that

$$\sum_{\nu=-\infty}^{+\infty} W_\nu \;=\; \mathrm{L}^2(\mathbb{R}),$$

and that $\{\psi_{\nu,l}(x) \mid l,\nu \in \mathbb{Z}\}$ is a Riesz basis of $\mathrm{L}^2(\mathbb{R})$. Since the mother wavelet is also an element of V_1, there exists a sequence $(g_k) \in \ell^2(\mathbb{Z})$ such that

$$\psi(x) \;=\; \sqrt{2}\sum_k g_k\,\varphi(2x \Leftrightarrow k). \tag{3}$$

If $\{\varphi_{\nu,l}(x) \mid l \in \mathbb{Z}\}$ is an orthonormal basis of V_ν, then we have an *orthogonal* multiresolution analysis. In this case we also choose $\{\psi_{\nu,l}(x) \mid \nu,l \in \mathbb{Z}\}$ to be an orthonormal basis of $\mathrm{L}^2(\mathbb{R})$. A *biorthogonal* multiresolution analysis is as above together with a dual multiresolution analysis $\{\tilde{V}_\nu\}$, i.e., the sets of functions $\{\tilde{\varphi}_{\nu,l}(x) \mid l \in \mathbb{Z}\}$ and $\{\tilde{\psi}_{\nu,l}(x) \mid l \in \mathbb{Z}\}$ are bases dual to $\{\varphi_{\nu,l}(x) \mid l,\nu \in \mathbb{Z}\}$ and $\{\psi_{\nu,l}(x) \mid l,\nu \in \mathbb{Z}\}$ respectively, i.e.,

$$\langle\, \varphi_{\nu,l}, \tilde{\varphi}_{\nu,l'} \,\rangle \;=\; \delta_{l-l'} \quad \text{and} \quad \langle\, \psi_{\nu,l}, \tilde{\psi}_{\nu',l'} \,\rangle \;=\; \delta_{\nu-\nu'}\delta_{l-l'} \quad \text{for} \quad \nu,\nu',l,l' \in \mathbb{Z}.$$

The dual scaling function and wavelet satisfy

$$\tilde{\varphi}(x) \;=\; \sqrt{2}\sum_k \tilde{h}_k\,\tilde{\varphi}(2x \Leftrightarrow k), \qquad \tilde{\psi}(x) \;=\; \sqrt{2}\sum_k \tilde{g}_k\,\tilde{\varphi}(2x \Leftrightarrow k), \tag{4}$$

and

$$\sqrt{2}\,\tilde{\varphi}(2x \Leftrightarrow k) \;=\; \sum_l h_{k-2l}\,\tilde{\varphi}(x \Leftrightarrow l) + \sum_l g_{k-2l}\,\tilde{\psi}(x \Leftrightarrow l). \tag{5}$$

Given the coefficients $\lambda_{\nu,l} = \langle\, f, \tilde{\varphi}_{\nu,l} \,\rangle$ of a function in V_ν, one can find its coefficients in the spaces $V_{\nu-1}$ and $W_{\nu-1}$ with decomposition formulae that can be derived using (4),

$$\lambda_{\nu-1,l} \;=\; \sum_k \tilde{h}_{k-2l}\,\lambda_{\nu,k}, \quad \text{and} \quad \gamma_{\nu-1,l} \;=\; \langle\, f, \psi_{\nu-1,l} \,\rangle \;=\; \sum_k \tilde{g}_{k-2l}\,\lambda_{\nu,k}.$$

The inverse step involves a reconstruction formula that can be derived from (5),

$$\lambda_{\nu,k} \;=\; \sum_l h_{k-2l}\,\lambda_{\nu-1,l} + \sum_l g_{k-2l}\,\gamma_{\nu-1,l}.$$

When applied recursively, these formulae define a transformation, the *fast wavelet transform.*[15,16]

The approximation properties of the wavelet expansion is related to the number of vanishing moments of the dual wavelet, denoted by N,

$$\int_{-\infty}^{+\infty} x^p\,\tilde{\psi}(x)\,dx \;=\; 0 \quad \text{for} \quad p = 0,\ldots,N \Leftrightarrow 1\,.$$

In general any polynomial with degree less than N can be written as a linear combination of the functions $\varphi(x \Leftrightarrow l)$ with $l \in \mathbb{Z}$. For a smooth function $f(x) \in \mathcal{C}^N$ the number of vanishing moments determines the order of convergence of the wavelet approximation through the relation,

$$\|f(x) \Leftrightarrow P_\nu f(x)\| \;=\; \mathcal{O}(h^N) \quad \text{with} \quad h = 2^{-\nu}, \tag{6}$$

where P_ν is the projection onto V_ν. The wavelet coefficients also provide characterization of other function spaces. Typical results in this direction is that if a function f is Hölder (Lipschitz) continuous of order $0 < \alpha < 1$, so that $|f(x+h) \Leftrightarrow f(x)| = \mathcal{O}(h^\alpha)$, then the wavelet coefficients have an asymptotic behavior like

$$\gamma_{\nu,l} \;=\; \mathcal{O}(h^{\alpha+1/2}) \quad \text{with} \quad h = 2^{-\nu} \tag{7}$$

and that $f \in \mathrm{L}^2(\mathbb{R})$ if and only if

$$\sum_{\nu,l} |\gamma_{\nu,l}|^2 < \infty \,.$$

3 WAVELETS ON AN INTERVAL

In this section we will present the construction, given in,[1] of a (bi-)orthogonal multiresolution analysis for a closed interval $[A, B]$. This construction involves the construction of multiscale spaces $V_\nu[A, B]$ and detail spaces $W_\nu[A, B]$ with properties analogous to those given in Section 2, as well as collections of basis functions $\{\varphi_{\nu,k}\}_k$ and $\{\psi_{\nu,k}\}_k$ spanning these spaces. In the case of a bi-orthogonal system we also need to construct the dual basis functions $\{\tilde{\varphi}_{\nu,k}\}_k$ and $\{\tilde{\psi}_{\nu,k}\}_k$ and the corresponding spaces $\tilde{V}_\nu[A, B]$ and $\tilde{W}_\nu[A, B]$.

The characteristic decay of the wavelet coefficients of a function f that belongs to one of the classical smoothness spaces (7), depends on the localization and size (in terms of certain function space norms) of the scaling functions $\varphi_{\nu,k}$ and wavelets $\psi_{\nu,k}$, as well as on the number of vanishing moments of the wavelets. The dual wavelet will have N vanishing moments precisely when $\mathcal{P}_{N-1} \subset V_\nu[A, B]$, where $\mathcal{P}_{N-1}$ is the set of polynomials of degree less than N. This must be satisfied by any construction of wavelets on an interval. The localization and size properties of the constructed bases are discussed at the end of Section 3.1.

3.1 Construction of wavelets on an interval

Fix a bi-orthogonal multiresolution analysis on the line, given by scaling functions Φ, $\tilde{\Phi}$ and wavelets Ψ, $\tilde{\Psi}$. Let $[A, B]$ be the closed interval under consideration, and define, for each scale ν, the following set of indices.

$$S_\nu \;=\; \{k : \operatorname{supp} \Phi_{\nu,k} \cap (A, B) \neq \emptyset\} \,.$$

This set is then divided into the following subsets.

$$\begin{aligned} S_{\nu,L} &= \{k : \Phi_{\nu,k} \text{ is supported near the left endpoint } A\} \\ S_{\nu,R} &= \{k : \Phi_{\nu,k} \text{ is supported near the right endpoint } B\} \end{aligned}$$

and

$$S_{\nu,I} \;=\; S_\nu \setminus \big(S_{\nu,L} \cup S_{\nu,R}\big) = \{k : \Phi_{\nu,k} \text{ is supported away from the endpoints}\} \,.$$

The sets $\tilde{S}_\nu$, e.t.c. are defined in a similar way. To avoid unnecessary complications we will only consider the case when the endpoints are independent in the sense that $S_{\nu,L} \cap S_{\nu,R} = \emptyset$. Let ν_0 be the smallest such integer. We also mention that it may be favorable from a stability point to make $S_{\nu,L}$ and $S_{\nu,R}$ larger while $S_{\nu,I}$ smaller than they actually have to be. Next we define, for $\alpha = 0, \ldots, N \Leftrightarrow 1$, the functions

$$\begin{aligned} (X \Leftrightarrow A)^\alpha_{\nu,L} &= \sum_{k \in S_{\nu,L}} \langle\, \tilde{\Phi}_{\nu,k}, 2^\alpha (x \Leftrightarrow A)^\alpha \,\rangle\, \Phi_{\nu,k} \;=\; \sum_{k \in S_{\nu,L}} (x \Leftrightarrow A)^\alpha_\nu(k)\, \Phi_{\nu,k} \\ (X \Leftrightarrow B)^\alpha_{\nu,R} &= \sum_{k \in S_{\nu,R}} \langle\, \tilde{\Phi}_{\nu,k}, 2^\alpha (x \Leftrightarrow B)^\alpha \,\rangle\, \Phi_{\nu,k} \;=\; \sum_{k \in S_{\nu,R}} (x \Leftrightarrow B)^\alpha_\nu(k)\, \Phi_{\nu,k} \end{aligned}$$

and set

$$V_\nu[A,B] \;=\; \mathrm{span}\,\{(X \Leftrightarrow A)^\alpha_{\nu,L}\}_{0\leqslant\alpha\leqslant N-1} \;\cup\; \mathrm{span}\,\{\Phi_{\nu,k}\}_{k\in S_{\nu,I}} \;\cup\; \mathrm{span}\,\{(X \Leftrightarrow B)^\alpha_{\nu,R}\}_{0\leqslant\alpha\leqslant N-1}.$$

It is now clear that we have achieved $\mathcal{P}_{N-1} \subset V_\nu[A,B]$. We refer to[1] for proof of the fact that the functions on the right-hand side of (8) forms a basis for $V_\nu[A,B]$ and that the spaces are nested. In fact $\{V_\nu[A,B]\}$ is a multiresolution analysis of $\mathrm{L}^2[A,B]$. The normalization factor 2^α has been chosen so that $(x \Leftrightarrow A)^\alpha_\nu(k)$ does not depend on the scale ν when A is a dyadic point.

Given a dual multiresolution analysis $\tilde{V}_\nu$ on the line we still have some freedom in choosing the spaces $\tilde{V}_\nu[A,B]$ If $\mathcal{P}_{N-1} \subset \mathrm{span}\{\tilde{\Phi}_{\nu,k}\}$, then we may proceed as above in the construction of the basis functions and spaces $\tilde{V}_\nu[A,B]$. This is however not always the case and the dual functions $\{\widetilde{(X \Leftrightarrow A)}^\alpha_{\nu,L}\}$ and $\{\widetilde{(X \Leftrightarrow B)}^\alpha_{\nu,R}\}$ have to be constructed in some other way. A choice that is particularly rewarding from a stability pointy of view is to have all the dual scaling functions and wavelets supported in the open interval (A,B), see[1] for a discussion of this case. It remains to find biorthogonal bases for the spaces $V_\nu[A,B]$ and $\tilde{V}_\nu[A,B]$. To this end define the matrices

$$M_L \;=\; \Big(\, \langle\, \widetilde{(X \Leftrightarrow A)}^\alpha_{\nu,L}, (X \Leftrightarrow A)^\beta_{\nu,L} \,\rangle_{\,\mathrm{L}^2[A,B]} \Big)_{0\leqslant\alpha,\beta\leqslant N-1}$$

and

$$M_R \;=\; \Big(\, \langle\, \widetilde{(X \Leftrightarrow B)}^\alpha_{\nu,R}, (X \Leftrightarrow B)^\beta_{\nu,R} \,\rangle_{\,\mathrm{L}^2[A,B]} \Big)_{0\leqslant\alpha,\beta\leqslant N-1}$$

and let $L * U = M_L$, $C * D = M_R$ be factorizations of these matrices. The collections

$$\varphi_{\nu k} \;=\; \begin{cases} \sum U^{-1}_{\alpha,\beta}(X \Leftrightarrow A)^\beta_{\nu,L} & \text{if } k = k_{\nu,L} \Leftrightarrow 1 \Leftrightarrow \alpha \text{ for } \alpha = 0,\ldots,N \Leftrightarrow 1, \\ \Phi_{\nu k} & \text{if } k_{\nu,L} \leqslant k \leqslant k_{\nu,R}, \\ \sum D^{-1}_{\alpha,\beta}(X \Leftrightarrow B)^\beta_{\nu,R} & \text{if } k = k_{\nu,R} + 1 + \alpha \text{ for } \alpha = 0,\ldots,N \Leftrightarrow 1, \end{cases} \tag{8}$$

and similarly for the dual functions, will then constitute biorthogonal bases of the spaces $V_\nu[A,B]$ and $\tilde{V}_\nu[A,B]$ respectively. Since $\varphi_{\nu,k} \in V_\nu[A,B] \subset V_{\nu+1}[A,B]$, there exist matrices H such that

$$\varphi_{\nu,k} \;=\; \sum_l H_{kl}\, \varphi_{\nu+1,l},$$

where H has the form,

$$H \;=\; \left(\begin{array}{c|c|c} H^{LL} & H^{LI} & 0 \\ \hline 0 & H^{II} & 0 \\ \hline 0 & H^{RI} & H^{RR} \end{array}\right).$$

The main part of H is the banded matrix $H^{II}_{kl} = h_{l-2k}$, while the remaining parts of H are small matrices that only involves indices close to the endpoints. For the dual functions we have,

$$\tilde{\varphi}_{\nu,k} = \sum_l \tilde{H}_{kl}\, \tilde{\varphi}_{\nu+1,l}$$

where $\tilde{H}$ has a form similar to H. We now turn to the construction of the detail spaces $W_\nu[A,B]$ and $\tilde{W}_\nu[A,B]$. These are given by the relations

$$V_{\nu+1}[A,B] \;=\; V_\nu[A,B] \;\dot{+}\; W_\nu[A,B]; \qquad W_\nu[A,B] \;\perp\; \tilde{V}_\nu[A,B]\,;$$

and similar relations for the space $\tilde{W}_\nu[A,B]$. The functions in the collection $\{\Psi_{\nu,k} : \Psi_{\nu k} \in \mathrm{span}\{\varphi_{\nu+1,m}\}_{m\in S_{\nu+1,I}}\}$ immediately qualify as a part of a basis for $W_\nu[A,B]$, i.e., they are in $V_{\nu+1}[A,B]$ and they are orthogonal to

$\tilde{V}_\nu[A,B]$. The remaining wavelets at the endpoints can now be built from those scaling functions $\varphi_{\nu+1,k}$ near the endpoints A and B that are not spanned by what we already have. A basis for $\tilde{W}_\nu[A,B]$ is built in the same way.

After necessary orthogonalizations and normalizations we obtain bases $\{\psi_{\nu,k}\}$ and $\{\tilde{\psi}_{\nu,k}\}$ for $W_\nu[A,B]$ and $\tilde{W}_\nu[A,B]$ respectively, that obey the desired biorthogonality relations. As for the scaling functions there exist refinement matrices G and $\tilde{G}$ such that

$$\psi_{\nu,k} \;=\; \sum_l G_{kl}\,\varphi_{\nu+1,l}; \quad \tilde{\psi}_{\nu,k} \;=\; \sum_l \tilde{G}_{kl}\,\tilde{\varphi}_{\nu+1,l}\,.$$

The matrices G and $\tilde{G}$ also have forms similar to H. As was discussed in the beginning of this section it is essential that the constructed scaling functions and wavelets on the interval preserve their localization and size properties. This is ensured if the corner matrices in the refinement relations have bounded dimension and norm. In[1] this was proved to hold at least for rational endpoints. It follows that the characterization of function spaces on the interval is as powerful as in the case of the real line. Characterizations like (7) still hold when one uses appropriate definition of the basis functions.

3.2 The fast wavelet transform

A function $f \in \mathrm{L}^2[A,B]$ can now be expanded in the following way

$$f \;=\; \sum_k \lambda_{\nu_0,k}\;\varphi_{\nu_0,k} + \sum_{\nu\geqslant\nu_0,k} \gamma_{\nu,k}\;\psi_{\nu,k}\,.$$

where the sum converges in L^2. We define the wavelet transform of f to be the sequence

$$WT(f) \;=\; \{\lambda_{\nu_0,k}\}_k \cup \bigcup_{\nu\geqslant\nu_0} \{\gamma_{\nu,k}\}_k\,.$$

Suppose now that $f = \sum \lambda_{\nu+1,k}\varphi_{\nu+1,k}$ is in $V_{\nu+1}[A,B]$ for some $\nu \geq \nu_0$. This function then splits into $f = f_\nu + g_\nu$ with

$$V_\nu[A,B] \ni f_\nu \;=\; \sum_k \lambda_{\nu k}\,\varphi_{\nu k}, \quad \text{and} \quad W_\nu[A,B] \ni g_\nu \;=\; \sum_k \gamma_{\nu k}\,\varphi_{\nu k},$$

This splitting corresponds to the transforms

$$\lambda_{\nu,k} \;=\; \sum_l \tilde{H}^\nu_{kl}\,\lambda_{\nu+1,l}, \quad \text{and} \quad \gamma_{\nu,k} \;=\; \sum_l \tilde{G}^\nu_{kl}\,\lambda_{\nu+1,l}.$$

The reconstruction formula is given by

$$\lambda_{\nu+1,l} \;=\; \sum_k H^\nu_{kl}\,\lambda_{\nu k} + G^\nu_{kl}\,\gamma_{\nu k}.$$

Due to the banded nature of the refinement matrices this turns into fast, $\mathcal{O}(N)$, algorithms for the wavelet transform and its inverse.

3.3 Recursive wavelets

Given the filter coefficients h_k and $\tilde{h}_k$, and the corresponding high-pass filters we may proceed as follows to construct a recursive multiresolution analysis for a finite scale ν_{max}. Let $\varphi = \tilde{\varphi} = \chi_{[01]}$ and define a bases for $V_{\nu_{max}[A,B]}$ and $\tilde{V}_{\nu_{max}[A,B]}$ by

$$\varphi^{\nu_{max}}_k \;=\; 2^{\nu_{max}/2}\varphi(2^{\nu_{max}}x \Leftrightarrow k) \quad \text{and} \quad \tilde{\varphi}^{\nu_{max}}_k \;=\; 2^{\nu_{max}/2}\varphi(2^{\nu_{max}}x \Leftrightarrow k).$$

For $\nu < \nu_{max}$ we define bases for $V_\nu[A,B]$ and $W_\nu[A,B]$ recursively through the relations

$$\varphi_k^\nu \;=\; \sum_k h_{m-2k}\,\varphi_m^{\nu+1} \quad \text{and} \quad \Psi_k^\nu \;=\; \sum_k g_{m-2k}\,\varphi_m^{\nu+1},$$

and similarly for the dual basis functions. When applying the above construction for wavelets on a closed interval to this recursively defined system we obtain what we call recursive wavelets for the interval. This construction has the advantage that sampling and preconditioning on the finest level ν_{max} become trivial. See[1] for a more thorough discussion of recursive wavelets.

4 SPLITTING AND MERGING

Suppose we are given a function f on the interval $[A,C]$ and we want to split it around a point B. We write

$$f = f^{left} + f^{right},$$

where we define $f^{left} = f \cdot \chi_{[A,B]}$ and $f^{right} = f \cdot \chi_{[B,C]}$. We now want a fast algorithm to go from $WT(f)$ related to the interval $[A,C]$, to $WT(f^{right})$ and $WT(f^{left})$ related to the subintervals $[A.B]$ and $[B,C]$ respectively. It is easy to see that almost all coefficients of $WT(f^{right})$ and $WT(f^{left})$ will correspond to coefficients of $WT(f)$. The only coefficients that change are the ones that correspond to wavelets whose supports are close to B. This means that the number of coefficients that needs to be recalculated is proportional only to the length of the filters and to the number of levels. The splitting algorithms thus only needs to recombine very localized coefficients and is a typical example of wavelet probing. The inverse algorithm is called merging.

5 SEGMENTATION

Given a data sequence $U = (u_i)_{0 \leqslant i < n}$ we want to split it into several subsequences $(u_i)_{s_j \leqslant i < s_{j+1}}$ for $0 \leqslant j \leqslant m$ where $s_j < s_{j+1}$, $s_0 = 0$ and $s_m = n \Leftrightarrow 1$. The splitpoints s_j are chosen according to a certain criterion. The general outline of the algorithm is: try every possible location and check whether putting a splitpoint at this location would be beneficial to the selected criterion. We use the recursive multiresolution analysis and let data sequence U correspond to a function $u(x)$ such that

$$u(x) \;=\; \sum_{i=0}^{n-1} u_i\,\varphi_i^0(x),$$

where $\varphi_i^0(x) = \chi_{[0,1]}(x \Leftrightarrow i)$. Note that the function $u(x)$ is supported on the interval $[0,n]$. We split the sequence U at an integer point i into two subsequences U_i^{left} and U_i^{right}.

We work here with compression based criteria. We consider it to be useful to put in a segmentation point when splitting the data at this point improves the compression ratio. Therefore we introduce a cost function $K(U)$ which measures how hard it is to compress the sequence U. A very simple cost function would be just to count the number of coefficients of the wavelet transform that are larger in absolute value than a certain threshold. Another cost function can be constructed by setting forth the percentage of coefficients one wants to encode. For each splitpoint one picks the so many largest coefficients of the transform. The cost function can then be defined as the norm of the difference between the original data and the data reconstructed from these coefficients. This relies on the equivalences of different norms as described in section 2. Note that there is also a cost associated with introducing a splitpoint as its location needs to be coded too. We denote this cost by $L(i)$. The overall cost of putting a splitpoint at i is then given by $C(i) = K(U_i^{left}) + K(U_i^{right}) \Leftrightarrow K(U) + L(i)$. Note that this cost function can be implemented efficiently because it involves only the coefficients associated with the splitpoint.

The segmentation points are chosen as local minima of the cost function. The minimum distance between two consecutive segmentation points is determined by the fact that they have to be independent in the sense of section 3.1. The algorithm can now be written as

for $i \leftarrow 1\,(1)\,n \Leftrightarrow 2$

split the interval into $[0, i]$ and $[i+1, n]$

$C(i) = K(U_i^{left}) + K(U_i^{right}) \Leftrightarrow K(U) + L(i)$.

end for

choose segmentation points such that $C(s_j)$ is a local minimum

compress each segment separately.

It can be implemented fast by using the splitting and merging algorithm. As we mentioned before the complexity of the algorithm is only $\mathcal{O}(n \log(n))$.

Segmenting has potential of reducing the number of coefficients needed to represent the data within a certain accuracy. The additional cost of coding the position of the splitpoints is not very high as can be seen from the following argument. When coding the coefficients of a wavelet transform it is not only important to code their size but also their position, i.e. their index ν, l. In fact the quality of the reconstructed data is very sensitive to the coding of the positions. This is related to the fact that the wavelet transform is not translation invariant, i.e. shifting the data samples by one yields a completely different wavelet transform. As a result the positions of the coefficients have to be coded exactly which is expensive in terms of compression. An advantage of the segmentation based compression is that the position of the splitpoints does not need to be coded very accurately. This can be understood by looking at the following example. Suppose we have a one dimensional piecewise constant signal with a jump discontinuity. The discontinuity corresponds to the optimal splitting point. Suppose we miss this splitting point by one sample. This means that one side will still be ok while the other one will have a spike at the first sample. This corresponds to a very local distortion with a very high frequency. At high ratio compression it is very likely to disappear. Remember that compression usually results in a smoothing of the data. The reconstructed signal is then again a piecewise constant function, the only difference being that the jump discontinuity is moved by one sample. In most applications this is hardly an objection as long as the discontinuity is not smoothed out. It is precisely the segmentation which will prevent the discontinuities from getting smoothed out. This means that, e.g. in an image, the edges can be replaced by smooth curves that can be coded efficiently.

For segmentation of speech signals it might be better to work with wavelet packets.[7,23] Speech typically consists of localized waveforms that can be compressed very efficiently with wavelet packets. The construction in section 3 can be used to define wavelet packets on closed sets.[9]

6 NUMERICAL RESULTS

In this section we give some numerical results of our segmentation algorithm. We use a computer program that does the splitting and merging for the biorthogonal piecewise linear wavelets. Note that in the interior of the the interval this corresponds to using the piecewise linear B-spline as scaling function and the function ${}_{2,2}\varphi$ in[6] as dual scaling function. Note that the filter coefficients used at the edges of the interval both depend on the level and on the endpoints. As cost function we use the number of coefficients larger than a certain threshold.

As first example we take a piecewise constant function. Figure 1 shows the function together with the cost function. The minima of the cost function exactly correspond to the discontinuities. Note that the minimum is very localized. Basically if you miss the splitting point by just one, all the benefit can vanish. Each segment can

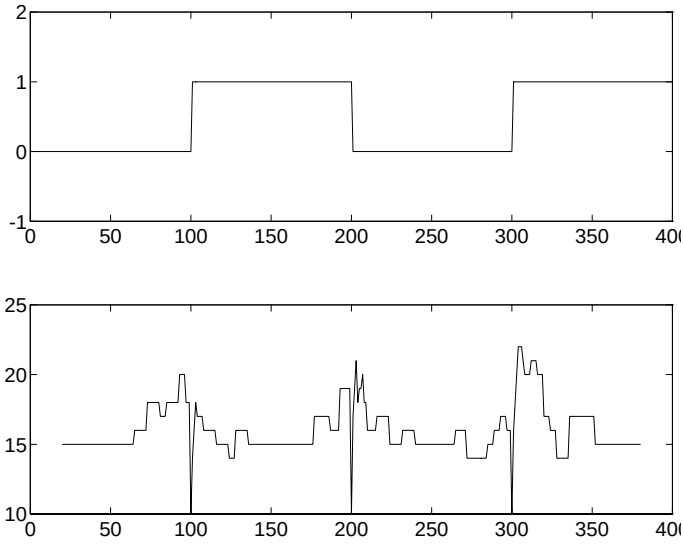

Figure 1: Piecewise constant data and cost function.

trivially be compressed since it is just a constant. Note that at each splitting point one wins about 15 coefficients. The power of the algorithm lies in the fact that the local gain accumulates when taking these five breakpoints as splitting points.

The second example is a piecewise linear function. This is a typical function that could result from the parametrization of a rectangle shaped edge in an image. The result is given in figure 2. Note that the minima of the cost function in this case are spread out over 2 locations. This is natural because the sample at the top can either go to the left or right segment. Each segment in this case is a linear and again can trivially be compressed.

These two examples are of course very easy because each segment can exactly be represented by only one or two coefficients. In the next example we take a look at a row out of the classical Lena image. The result is given in figure 3. The minima correspond to locations that one expects visually.

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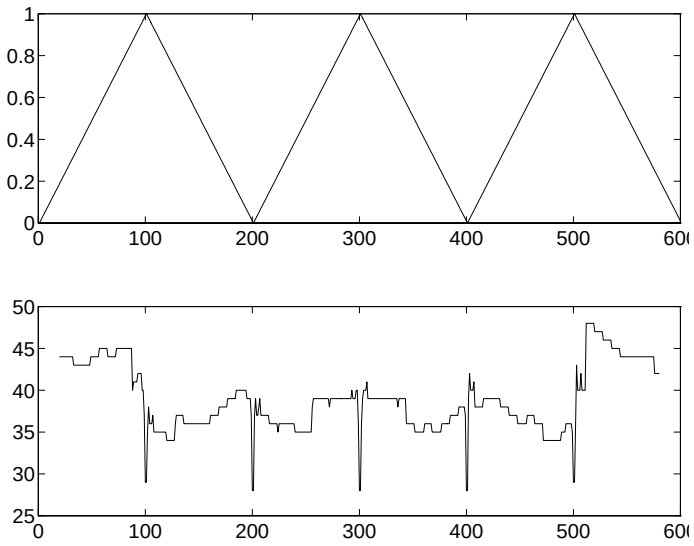

Figure 2: Piecewise linear data and cost function.

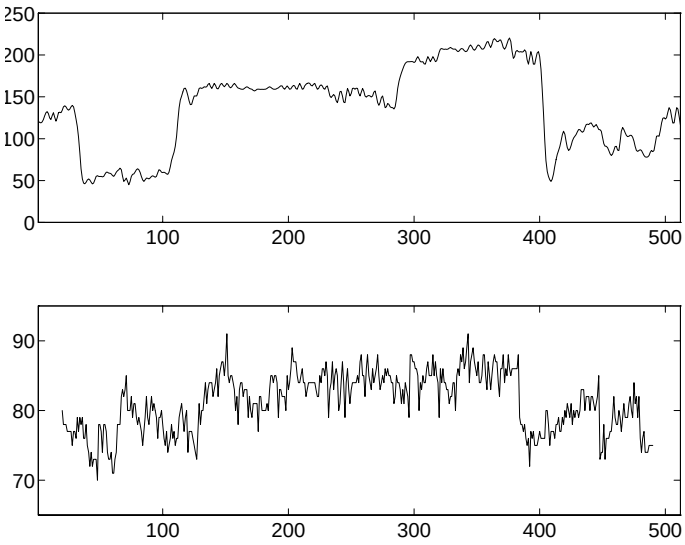

Figure 3: Row of Lena image and cost function.

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