COMPACTLY SUPPORTED WAVELETS WHICH ARE BIORTHOGONAL WITH RESPECT TO A WEIGHTED INNER PRODUCT

Wim Sweldens*

Department of Mathematics, University of South Carolina, Columbia SC 29208
Department of Computer Science, Katholieke Universiteit Leuven, Belgium
sweldens@math.scarolina.edu

1. Introduction. The basic idea of wavelets and multiresolution analysis is to use the dyadic translates and dilates of one function as a basis of L_2 [4, 5, 8]. Wavelets are often constructed with the use of the Fourier transform. The reason is that translation and dilation on the frequency side become algebraic operations. We call these wavelets *algebraic wavelets*. Examples of compactly supported algebraic wavelets are Daubechies' orthogonal wavelets and spline wavelets [3, 5]. Algebraic wavelets are (bi)orthogonal with respect to the L_2 inner product. In this paper we show how to construct wavelets adapted to a weighted inner product. We call such wavelets *weighted wavelets*.

2. Weighted multiresolution analysis. Consider a locally integrable and positive weightfunction w and the weighted inner product

$$\langle f, g \rangle_w = \int_{-\infty}^{+\infty} w(x)\, f(x)\, \overline{g(x)}\, dx.$$

Such an inner product can e.g. result from the parametrization of an inner product on a Lipschitz curve.

It is obvious that in the case of weighted wavelets, the algebraic structure can no longer be used. We thus adapt the definition of multiresolution analysis. Let us first concentrate on which properties are fundamental for the wavelets to be a powerful tool. Essentially they are the following:

1. Explicit expression for their coordinate functionals exist (through the dual wavelets).
2. They have compact support.
3. They have vanishing moments.
4. They are smooth.
5. Fast transforms are available.

Because of property (2) and (3), the wavelets are localized in space and frequency. Consequently, the wavelet coefficients of a function decay rapidly where the function is smooth. By setting the small coefficients to zero, one can accurately represent a function with only a few wavelet functions. This is the key to applications in data compression and numerical analysis. Property (4) is of importance to obtain convergence in a smoothness norm.

We define a multiresolution analysis as a sequence of closed subspaces $V_j \subset L_2$ so that

1. $V_j \subset V_{j+1}$,
2. $\bigcup_{j=-\infty}^{+\infty} V_j$ is dense in L_2 and $\bigcap_{j=-\infty}^{+\infty} V_j = \{\mathbf{0}\}$,
3. Scaling functions $\varphi_{j,k}$ exist so that $\{\varphi_{j,k}\}_k$ is a Riesz basis of V_j.

This implies that for every scaling function $\varphi_{j,k}$, coefficients $\{h_{j,k,l}\}$ exist, so that it satisfies a refinement relation

$$\varphi_{j,k} = \sum_l h_{j,k,l}\, \varphi_{j+1,2k+l}. \tag{1}$$

Each scaling function satisfies a different refinement relation. The dual multiresolution analysis consists of spaces $\widetilde{V}_j$ with bases generated by dual scaling functions $\widetilde{\varphi}_{j,k}$ that are biorthogonal with the scaling functions,

$$\langle \varphi_{j,k}, \widetilde{\varphi}_{j,k'} \rangle_w = \delta_{k-k'}. \tag{2}$$

The dual scaling functions satisfy refinement relations with coefficients $\{\widetilde{h}_{j,k,l}\}$. Note that the coefficients of the refinement relation can be written as

$$h_{j,k,l} = \langle \varphi_{j,k}, \widetilde{\varphi}_{j+1,2k+l} \rangle_w .$$

Define $N-1$ to be the highest degree of polynomials that can be represented as a linear combination of the $\{\varphi_{j,k}\}_k$, and similarly for $\widetilde{N}$.

We define the space W_j to be a complement of V_j in V_{j+1}, and assume $\{\psi_{j,k}\}_k$ is a Riesz basis for this space. We have the refinement relation

$$\psi_{j,k} = \sum_l g_{j,k,l}\, \varphi_{j+1,2k+l}, \tag{3}$$

and similarly for the dual wavelets. The dual wavelets $\widetilde{\psi}_{j,k}$ are biorthogonal to the wavelets, or

$$\langle \psi_{j,k}, \widetilde{\psi}_{j',k'} \rangle_w = \delta_{k-k'}\, \delta_{j-j'}.$$

The wavelets (resp. dual wavelets) have $\widetilde{N}$ (resp. N) vanishing weighted moments.

We want to construct compactly supported basis functions and dual functions, where somehow the index j corresponds to scale and the index k to location. This is true if a closed interval $I \subset \mathbf{R}$ exists such that $\operatorname{supp} \varphi_{j,k} \subset 2^{-j}(I+k)$, and similarly for the wavelets and dual functions. This implies that the coefficient sequences of the refinement relations are finite. A fast wavelet transform, between the coefficients $\lambda_{n,l}$ of a function of V_n and its wavelet coefficients $\gamma_{j,l}$ on the coarser levels $(j<n)$, recursively uses the relations

* Research Assistant of the National Fund of Scientific Research Belgium and partially supported by NSF EPSCoR Grant EHR 9108772.

$$\lambda_{j,k} = \sum_l \widetilde{h}_{j,k,k-2l}\, \lambda_{j+1,l}, \quad \gamma_{j,k} = \sum_l \widetilde{g}_{j,k,k-2l}\, \lambda_{j+1,l},$$

and

$$\lambda_{j+1,k} = \sum_l h_{j,l,k-2l}\, \lambda_{j,l} + \sum_l g_{j,l,k-2l}\, \gamma_{j,l}.$$

All filter sequences are finite. The only difference with the algebraic fast wavelet transform is that the filter coefficients are different for every coefficient. Note that with this setting, we can satisfy properties (1)–(5) without the use of the algebraic structure.

Examples. A simple example of wavelets that are orthogonal with respect to a weighted inner product exists; it is a variant of the Haar wavelets. Consider the multiresolution analysis where V_j is the space of functions that are piecewise constant on dyadic intervals of length 2^{-j}. Obviously, the set $\{\varphi_{j,k}\}_k$ with $\varphi_{j,k} = \chi_I$ and $I = [k2^{-j}, (k+1)2^{-j})$ forms an orthogonal basis of V_j for any weight function. Let $\psi_{j,k}$ be a function that is piecewise constant on the intervals $[2^{-j}k, 2^{-j-1}(2k+1))$ and $[2^{-j-1}(2k+1), 2^{-j}(k+1))$, zero elsewhere, and with one vanishing weighted moment. It then follows immediately that the set $\{\psi_{j,k}\}$ is orthogonal with respect to the weighted inner product. Since the wavelet is not symmetric, we call it the *unbalanced Haar wavelet*. A clever generalization of this idea to multiple dimensions was recently found by Marius Mitrea [9]. The unbalanced Haar wavelets and the Mitrea wavelets have as disadvantage that they only have one vanishing moment and that they are non-smooth.

Several other constructions of weighted wavelets already exist, see [1, 2, 8, 12]. However, they have the disadvantage that the wavelets are not compactly supported.

In this paper we construct smooth, compactly supported, weighted wavelets with more vanishing moments. First of all, we use a more restrictive condition on the compact support than the one suggested above, in the sense that $\operatorname{supp} \varphi_{j,k} = 2^{-j}(\operatorname{supp} \varphi_{0,0} + k)$, and similarly for the wavelets and dual functions. This implies that the index range (over l) of the non-zero coefficients in the refinement relations (and thus the fast wavelet transform) is independent of j and k.

Secondly, we fix the dual scaling functions and construct scaling functions, wavelets and dual wavelets with the desired properties. More precisely, we let the dual scaling functions be the indicator functions on the dyadic intervals, $\widetilde{\varphi}_{j,k} = \chi_I$ with $I = [k2^{-j}, (k+1)2^{-j})$. The refinement relation for the dual scaling functions is

$$\widetilde{\varphi}_{j,k} = \widetilde{\varphi}_{j+1,2k} + \widetilde{\varphi}_{j+1,2k+1}. \tag{4}$$

From the refinement relations (1) and (4) and the biorthogonality (2) it follows that

$$h_{j,k,2l} + h_{j,k,2l+1} = \delta_l. \tag{5}$$

3. Construction. Since the algebraic structure is gone, the Fourier transform cannot be used in the construction. Our construction is a generalization of the average-interpolation scheme of David Donoho [6]. This is a subdivision scheme for the construction of biorthogonal algebraic wavelets without the use of the Fourier transform.

From symmetry arguments one can understand that N has to be odd. We let $N = 2D+1$. Suppose one wants to synthesize a function of V_i,

$$f = \sum_k \lambda_{i,k}\, \varphi_{i,k}. \tag{6}$$

The idea of a subdivision scheme is to write this function in the basis of a finer scale space V_j with $j > i$, and to let j tend to infinity.

One step of the subdivision scheme consists of, given the coefficients $\lambda_{j,k}$ on one level, calculating the coefficients on the next finer level $\lambda_{j+1,k}$. For each group of N coefficients $\{\lambda_{j,k-D}, \ldots, \lambda_{j,k}, \ldots, \lambda_{j,k+D}\}$, it involves two steps:

1. Construct a polynomial P of degree N so that
$$\langle P, \widetilde{\varphi}_{j,k+l} \rangle_w = \lambda_{j,k+l} \quad \text{for} \quad -D \leqslant l \leqslant D.$$
2. Calculate two coefficients on the next finer level as $\lambda_{j+1,m} = \langle P, \widetilde{\varphi}_{j+1,m} \rangle_w$, with $m = 2k, 2k+1$.

This defines a subdivision operator $\mathcal{U}_j$ with

$$\{\lambda_{j+1,k}\}_k = \mathcal{U}_j\, \{\lambda_{j,k}\}_k.$$

One now finds the coefficients $h_{j,k,l}$, which are needed in the fast wavelet transform, by letting one of the $\lambda_{j,k}$ be equal to one and all others equal to zero, or, more precisely

$$\{h_{j,k,l}\}_l = \mathcal{U}_j\, \{\delta_{k-l}\}_l.$$

For each j and k, there are $2N$ non-zero coefficients $h_{j,k,l}$, namely the ones with $l = -N+1, \ldots, N$. The only information needed from the weightfunction to implement this construction on a computer are the local moments,

$$M^p_{j,k} = \langle x^p, \widetilde{\varphi}_{j,k} \rangle_w .$$

To synthesize f, define the following series of functions $(j \geqslant i)$,

$$f^{(j)} = \sum_k \lambda_{j,k}\, \widetilde{\varphi}_{j,k} / M^0_{j,k}.$$

If $\lim_{j\to\infty} f^{(j)}$ converges uniformly, we define f to be the limit function. This way we give meaning to the formal expression (6). In order to construct the scaling function $\varphi_{i,k}$, we start from the Kronecker sequence $\{\lambda_{i,l}\}_l = \{\delta_{k-l}\}_l$.

In the unweighted case, this scheme converges to an algebraic biorthogonal scaling function [6].

The idea to prove the convergence in the weighted case is to examine the behavior of the difference with the unweighted case. For a detailed treatment, we refer to [11]. We first show that the coefficients $h_{j,k,l}$ converge to the coefficients h_l from the unweighted case if j tends to infinity. This way we know that on a sufficiently fine level the polynomial P in the construction always exists. With a mild condition on the weight function, along the lines of a bounded oscillation condition, one can show that the convergence of the subdivision is uniform and the scaling functions are smooth. We show that even if the weight has large jump discontinuities, the scaling functions are still smooth.

4. Properties. The properties of the weighted scaling functions are summarized in the following theorem.

THEOREM 4.1. *The weighted scaling functions satisfy*
1. $\operatorname{supp}\varphi_{j,k} = 2^{-j}[-N+1+k, N+k]$,
2. $\int_{-\infty}^{+\infty} w(x)\,\varphi_{j,k}(x)\,dx = 1$,
3. $\langle\, \varphi_{j,k}, \widetilde{\varphi}_{j,k'}\,\rangle_w = \delta_{k-k'}$,
4. $\sum_k M_{j,k}^p\,\varphi_{j,k} = x^p$ *for* $0 \leqslant p < N$.

The following theorem shows how to find the wavelets and summarizes their properties.

THEOREM 4.2. *Assume that the scaling functions and the dual scaling functions are given as above. Choose the wavelet and dual wavelet as* $\psi_{j,k} = \varphi_{j+1,2k} - \varphi_{j+1,2k+1}$, *and*

$$\widetilde{\psi}_{j,k} = \sum_l g_{j,k,l}\,\widetilde{\varphi}_{j+1,2k+l},$$

with $g_{j,k,l} = (-1)^l\, h_{j,k+\lfloor l/2\rfloor,1-l}$. *Then*
1. $\langle\, \widetilde{\varphi}_{j,k}, \psi_{j,k'}\,\rangle_w = 0$ *and* $\langle\, \varphi_{j,k}, \widetilde{\psi}_{j,k'}\,\rangle_w = 0$,
2. $\langle\, \psi_{j,k}, \widetilde{\psi}_{j',k'}\,\rangle_w = \delta_{j-j'}\,\delta_{k-k'}$,
3. $\int_{-\infty}^{+\infty} w(x)\,\psi_{j,k}(x)\,dx = 0$,
4. $\int_{-\infty}^{+\infty} w(x)\,x^p\,\widetilde{\psi}_{j,k}(x)\,dx = 0$, *for* $0 \leqslant p < N$,
5. $\operatorname{supp}\psi_{j,k} = \operatorname{supp}\widetilde{\psi}_{j,k} = 2^{-j}[-D+k, D+1+k]$.

The vanishing moment property implies that if a function f belongs to $\mathcal{C}^N$, then the weighted wavelet coefficients decay as $\langle\, f, \widetilde{\psi}_{j,k}\,\rangle_w = \mathcal{O}(2^{-j(N+1)})$.

5. Applications. The weighted functions can be used for the rapid numerical solution of ordinary differential equations with boundary conditions. We consider the operator $\mathcal{L} = -D\,pD$, where p is bounded away from zero. The construction is based on the observation that

$$\langle\, \mathcal{L}\, f, g\,\rangle = \langle\, p\,D\,f, p\,D\,g\,\rangle_w,$$

where the weight is taken to be $w = 1/p$. Now consider biorthogonal, compactly supported, weighted wavelets $\psi_{j,k}$ and $\widetilde{\psi}_{j,k}$. Define operator wavelets as

$$\Psi_{j,k} = D^{-1}\,\psi_{j,k}/p,$$

and similarly for the dual wavelets. It then follows that

$$\langle\, \mathcal{L}\,\Psi_{j,k}, \Psi^*_{j',k'}\,\rangle = \delta_{j-j',k-k'}.$$

The operator wavelets diagonalize the operator. A similar observation was made independently in [2]. The operator wavelets here are compactly supported, since the weighted wavelets have at least one vanishing weighted moment. The fast wavelet transform associated with the operator wavelets is easy to implement, because of their compact support. This results in a linear, non-iterative algorithm for the numerical solution of ODEs [7]. It outperforms finite element and finite difference methods.

A similar idea is possible for constant coefficient operators such as the Helmholtz operator in one dimension. For these operator the wavelet method outperforms spectral methods.

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